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APPENDIX.

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PHILOSOPHICAL TRANSACTIONS.

XII. *An Account of Meteorological Observations in Four Balloon Ascents, made under the direction of the Kew Observatory Committee of the British Association for the Advancement of Science. By* JOHN WELSH, *Esq. Communicated by Colonel* SABINE, *R.A., Treas. and V.P.R.S., by the request of the Council of the British Association for the Advancement of Science.*

Received April 27,—Read May 26, 1853.

IN July 1852, the Committee of the Kew Observatory resolved to institute a series of balloon ascents, with the view of investigating such meteorological and physical phenomena as require the presence of an observer at a great height in the atmosphere. The arrangements made for carrying out this resolution have been stated by the Committee in their report to the Council of the British Association, a short account being at the same time given of some of the results derived from the ascents already made. Having been to a great extent entrusted by the Committee with the conduct of the observations and with the instrumental arrangements, I now, at their request, proceed to give a more detailed statement of the mode in which the experiments have been made, and of such results as may most readily be deduced from the observations recorded in the ascents.

The object to which especial attention was devoted, was the determination of the temperature and hygrometric condition of the air at different elevations above the earth's surface. Besides this, the observers were furnished with the means of procuring specimens of the air at different heights for the purpose of analysis, and of examining, if opportunity offered, whether the light reflected from the upper surface of the clouds was polarized.

§ 1. *Instruments and Apparatus.*

The instruments required for the investigations contemplated were—a barometer; dry and wet thermometers; an aspirator; REGNAULT's condensing hygrometer; DANIELL's dew-point hygrometer; a polariscope; and glass tubes, furnished with

stopcocks, from which the air had been exhausted. All the instruments which were at all liable to accident were supplied in duplicate. The construction of the meteorological instruments was confided by the Committee to Mr. P. Adie of London, under my own general superintendence. They were executed by him in a very satisfactory manner, having been made with much accuracy and with an anxious wish to promote the success of the experiments; many of the mechanical arrangements for the convenience of observation having also been devised by him.

Barometers.—The barometer employed was of the siphon form, on the construction generally known as Gay-Lussac's. The tube was affixed to a brass scale in much the same way as a thermometer is attached to its scale. The brass scale was fixed within a stout rosewood frame furnished with a door which could be closed during carriage. The diameter of the tube was 0·25 inch. The graduation was made from the middle point upwards and downwards; each division being $\frac{1}{10}$th of an inch long, but representing twice that value; so that an observation of either branch of the siphon would give the length of the column of mercury, subject to a correction for inequality of the tube and error in the position of the zero-point of the scale. A complete observation of the instrument required however readings of both branches of the siphon, the true height of the mercury being the mean of the two. In order to facilitate rapidity of observation, verniers were dispensed with, the height of the mercury being merely estimated with reference to the scale placed behind it, just as if it had been a thermometer of large calibre. As it would have been nearly impossible to obtain in the car of the balloon a complete reading of both branches of the siphon for each observation, the corrections to the readings of the upper branch alone were previously obtained, throughout the anticipated range of the mercury, by the help of a large vacuum apparatus at the Kew Observatory, which has been employed in the pendulum experiments of Colonel Sabine and Professor Stokes. The barometers having been suspended within the receiver, the air was exhausted by about half an inch of pressure at a time, and readings taken from which tables of corrections were computed for different heights of the mercury. These corrections have been applied to all the observations. The difference between the indications of the siphon barometers and those of the Kew standard was also observed: both barometers were found to read 0·025 inch *higher* than the standard. It was found, by intercomparisons made last year, that the standard barometer at the Royal Observatory, Greenwich, reads *lower* than the Kew standard by 0·003 inch. The balloon barometers thus read 0·028 inch *higher* than the Greenwich standard; and, as that barometer has been generally referred to in the computations of height, the equation +0·028 has been applied to the terrestrial observations to render them comparable with those of the balloon barometers. Each barometer was provided with a thermometer to indicate the temperature of the mercury. In order to obtain this temperature more accurately, the bulb of the thermometer (which was cylindrical, about $1\frac{1}{2}$ inch long and $\frac{1}{8}$th of an inch diameter) was immersed in mercury contained in a

tube of the same diameter as that of the barometer. The necessity for this precaution was found to be great, as very large differences sometimes existed between the temperature of the air thermometer and that of the mercury.

Dry and Wet Thermometers.—Two pairs of dry and wet thermometers were employed. One pair was mounted with the bulbs protected from radiation by a double conical shade, having highly polished silver surfaces, open at top and bottom to allow the circulation of the air. The inner shade was 2 inches high, $1\frac{3}{4}$ inch wide at the lower, and half an inch at the upper end: the outer shade was also 2 inches high, $2\frac{3}{4}$ inches wide at the lower, and $1\frac{3}{8}$ inch at the upper end*. Both thermometers were furnished with shades exactly similar; the bulbs being thus in the same circumstances, and completely protected from direct radiation. The thermometers were supported, $3\frac{1}{2}$ inches apart, by the arms of a light brass frame, also with a polished silver surface. A small brass cistern was fixed near the wet thermometer, from which water was conveyed to the bulb by a conducting string of floss silk; when however the temperature fell below the freezing-point, the string was cut away and the bulb occasionally dipped in water.

As it was of essential importance that the thermometers should acquire with the utmost possible rapidity the temperature of the surrounding air, an arrangement was made, in connection with the second pair of dry and wet thermometers, for producing artificially a more rapid current over the bulbs than they would be exposed to by the mere vertical motion of the balloon. It was also thought desirable to avoid any tendency to a stagnation of the vapour of water in the neighbourhood of the wet bulb owing to the want of a sufficient circulation of air to carry it off, as might be the case when the balloon was nearly stationary or moving very slowly. An increased velocity in the circulation of the air would also tend to remove the effects of radiation, if the thermometers were not already sufficiently protected by the shades. With these objects the following contrivance was adopted. The thermometers were fixed vertically with their bulbs enclosed in two tubes placed side by side, and connected with each other by a cross tube joining their upper ends; these tubes having silver surfaces, and being further protected by a silver shade of the same dimensions as the outer shade of the other pair of thermometers. The first tube, in which was the bulb of the dry thermometer, had at its lower end a communication with the air: by means of an aspirator a current was produced from this opening, upwards over the dry bulb, then passing, by the communication at the top, into the second tube down which it moved over the wet bulb, leaving it by an opening connected by a flexible pipe with the aspirator. By this means, the temperature of the air was determined in its passage over the dry bulb, and afterwards its hygrometric condition on coming in contact with the wet; the vapour of water formed at the latter being carried off immediately into the aspirator. The whole distance which the air had to

* It might have been preferable to make the inner shade cylindrical instead of conical, as the air would have circulated more freely.

travel, between its entrance into the tubes and its leaving the wet thermometer, was about 4 inches: the diameter of the tubes enclosing the bulbs was 0·4 inch, and that of the connecting tube 0·25 inch. The aspirator was a cylindrical bellows; the valves being so arranged that, when the aspirator was forced open, the air could only enter it by passing over the thermometers: it was worked by attaching a weight to the lower end which pulled it open, the upper end being fixed; when it had opened to nearly its full extent, it was closed by means of a cord passing over pulleys and drawn up by the hand; a large valve allowing the air to escape rapidly from the aspirator as it was closed, and a second valve preventing the air from being driven backwards over the thermometers. Care was taken, in the construction of the different parts, that the aperture of the tubes should not be smaller between the thermometers and the external air, than between them and the aspirator; otherwise the air might, by undergoing a certain degree of expansion, have come in contact with the bulbs in different conditions with respect to temperature and capacity for moisture from those of the external air. This was guarded against by applying a stopcock near the aspirator, whose aperture was sufficiently small. A second flexible tube, with a stopcock, connected the aspirator with Regnault's hygrometer; so that the same aspirator might be used simultaneously for both instruments. Two different sizes of aspirator were used in the different ascents; the one being 12 inches diameter, and extending to about 18 inches, occupying about $1\frac{1}{2}$ minute in its descent; the other was 9 inches diameter, extending to 12 inches in 30 to 40 seconds. This was sufficient to produce a current of air over the bulbs at the rate of 12 to 14 feet in a second; the vertical velocity of the balloon seldom exceeding 4 or 5 feet. The thermometers employed were of great sensibility; the bulbs being cylindrical, the diameter not exceeding $\frac{1}{13}$th of an inch, and the length varying from a half to three-quarters of an inch. The length of one degree of the scale was from $\frac{1}{25}$th to $\frac{1}{10}$th of an inch, so that they could readily be read by estimation to 0°·1. The graduation extended to 30° or 40° below zero of Fahrenheit. The scales of those used in the first ascent were of brass, but afterwards of ivory, in order to render the column of mercury more visible. The errors of all the thermometers were determined throughout the scale, from about 0° to 70°, by comparison with standards at the Kew Observatory; the comparisons below the freezing-point being made in mixtures of ice and salt. The corrections have been applied to all the readings. These thermometers were found to acquire the temperature of the air very rapidly: when heated 20° above the temperature of the air, and allowed to cool *at rest* in a confined room, they returned to within 0°·5 of the previous reading in about 100 seconds; when gently fanned, by being carried through the room at the rate of 5 or 6 feet in a second, they returned to within 0°·5 in 40 seconds; when under the action of the aspirator they returned to within 0°·5 in 30 seconds, and exactly to the original reading in 45 seconds. Any correction on account of sluggishness in the thermometers must thus be very small: this is shown by the observations of October 21, when the descent took place with about the same

velocity as the ascent, and observations were continued to within 3000 feet of the earth. The differences of temperature at the same height are scarcely appreciable, and even frequently in the opposite direction from what would result from insensibility in the thermometer. A few observations were taken during the *descent* on August 17, which, when compared with those made at the same height in the *ascent*, show a difference of about four degrees. The rapidity of the descent was on that occasion about twice as great as that of the ascent, which was also considerable, and the thermometers were not under aspiration. The protection from radiation has been examined by observing the thermometers within a room when alternately exposed to strong sunshine and shade—the effect upon the aspirated thermometers did not exceed 1°·5: in the open air, and with a gentle breeze, the effect was considerably less. The effect upon the free thermometer appeared to be greater; and the difference between its indications and those of the aspirated thermometer during some of the ascents is probably to be ascribed to this cause. It would appear from some portions of the series of August 26, that the long-continued exposure to the sun, in a nearly calm air, has produced an appreciable effect upon the readings of the thermometer, whether aspirated or free. Fortunately, with the exception of the ascent of August 26, the sun's radiation was never powerful; whilst on August 17, when the free thermometers were alone observed, the sun was scarcely ever visible. When the radiation was feeble, and the vertical motion of the balloon considerable, the two thermometers agreed very closely*. As hygrometers there is probably less difference in their value. In the examination of the results of the temperature observations, I have been led to prefer the indications of the aspirated thermometer.

Regnault's *Condensing Hygrometer*†.—The only difference in the construction of this instrument, from that usually adopted by M. Regnault, was that the small tube, by which the air enters the reservoir to agitate the ether, had a funnel-shaped opening at top to facilitate the supply of ether. The bulb of the thermometer was cylindrical, $1\frac{1}{2}$ inch long, and $\frac{1}{10}$th of an inch diameter. The scale was of ivory, and the thermometer was fixed into the reservoir by a cork.

No use was made of Daniell's hygrometer as that of M. Regnault was found much more convenient for such observations, being to a great extent self-acting.

Polariscope.—This instrument was supplied by Mr. Darker of Lambeth. Its principle is the same as that employed in Mr. Wheatstone's "Polar Clock;" the parts of the polariscope used having, I believe, actually formed portions of one of those instruments. It exhibits the existence of polarization in a conspicuous manner.

Exhausted Tubes for Collecting Air.—These tubes, which were constructed by Messrs. Negretti and Zambra, were about 9 inches long and $\frac{3}{4}$ inch diameter, fitted

* It would be advisable in any future experiments to apply additional shades to all the thermometers, and if possible to use a larger general screen at some distance from them.

† This instrument is described in the "Annuaire Météorologique de la France" for 1849, p. 221.

with stopcocks. They were prepared by Dr. MILLER previously to each ascent, and hermetically sealed immediately after their return to King's College.

§ 2. *Observing Arrangements, Personal and Instrumental.*

It was deemed advisable that, in the first ascents at least, two observers should take part in the work. Mr. R. B. NICKLIN, who for upwards of two years had been employed at the Kew Observatory and been practised in the observation of instruments, acted as my assistant in the first two ascents. Mr. NICKLIN's aid was of essential service, and I wish to express my acknowledgement for the careful manner in which he took the observations with which he was entrusted, and for the readiness with which he assisted me on several occasions, sometimes at considerable personal inconvenience, when unforeseen difficulties arose. Having in these two ascents acquired experience in the observations, and having got the instruments into better working order, in the last two ascents I undertook the observations alone, thus obtaining the power of reaching a greater elevation.

The car attached to the balloon was an oblong basket of wicker-work, about 6 feet long, 3 feet wide, and $2\frac{1}{2}$ feet deep. One end of this was occupied by the observers with the instruments, and the other by Mr. GREEN, who managed the balloon. A light board, a foot wide, was fixed across the car in front of the observers: at the extremity of this board, and projecting nearly a foot over the side of the car, was erected a light horizontal bar of wood, raised about 9 inches above the board, and inclined at an angle of about 45° to its length, the board being cut away beyond the bar so as not to present any resistance to the circulation of the air. Upon the bar were fixed the thermometers and hygrometers. The aspirator was fixed to the lower side of the board, in which a hole was cut to admit the connecting tubes. On the first ascent the barometers were suspended from the hoop by which the car is attached to the netting of the balloon; this was however found to be inconvenient; and in the subsequent ascents they were suspended by gimbals from the cross board, their verticality being secured by weights attached to the lower ends of the cases. When seated in the car for observation, Mr. NICKLIN occupying the right-hand corner and I the left, the stand supporting the thermometers was to my left, at a distance of about 18 inches; the aspirator being underneath the board, which served as a table before us: one barometer was immediately in front of Mr. NICKLIN, and the other before myself; the observations could thus be readily taken without rising from our places.

In order to obtain as continuous a record as possible of the variations of temperature and humidity, the observations were taken at very short intervals, generally at every minute, but frequently twice in a minute. In the first two ascents Mr. NICKLIN observed one barometer, whilst I observed the thermometers and hygrometers, taking an occasional observation of the second barometer as a check upon the indications of the other. A watch which had been set to Greenwich time was placed in sight of both observers. In the last two ascents, when I was the only observer, the

barometer was always read first, and immediately afterwards the thermometers and hygrometers; the whole time occupied being only a few seconds, the error, arising from the observations not being strictly simultaneous, must be very small. Note-books were provided with columns ruled and headed for the different instruments.

§ 3. *Circumstances of the Ascents and General Observations.*

The ascents were made with Mr. C. GREEN's large balloon, well known by the name of the "Royal Nassau." It has been fortunate, for the success of these experiments, that the Kew Committee obtained the cooperation of Mr. GREEN, whose pre-eminence as a skilful aëronaut has been established by upwards of 500 ascents; and whose control over his balloon is so complete, that no one who accompanies him can be otherwise than relieved from apprehension, and free to devote his attention calmly to the work before him. Mr. GREEN on all occasions showed the most anxious desire to contribute to the success of experiments, in which he took great interest.

The ascents took place from the Royal Vauxhall Gardens, which were liberally placed at the disposal of the Committee by Mr. WARDELL, the Lessee. The balloon was inflated with carburetted hydrogen gas, obtained from the Vauxhall and Phœnix Gas-works.

First Ascent, August 17.—The weather, previously to the 17th, had been somewhat variable; on the 16th the wind changed from S.E. to S.W., and on the day of ascent it was from south. Clouds covered about three-fourths of the sky, the lowest stratum being a few detached masses of loose cumulus; a dense mass of cirrostratus (or stratus) being above, with perhaps occasional patches of cumulus intermediate. The ascent commenced at $3^h\ 49^m$ P.M., after considerable difficulty had been experienced in the preliminary arrangements, owing to the force of the wind. A short time was lost at first in the attempt to put the instruments into more convenient order, and also from the novelty of the situation. The lowest clouds, which extended only over a small area, and were not near the balloon, were passed before they were noticed; their height was estimated at about 2500 feet. Between this height and about 13,000 feet, the air seemed free of clouds; after this, although the balloon was never in actual cloud, there seemed to be occasional masses of loose cumulus at no great distance. When at the greatest elevation, there was, at apparently a short distance above us, a thick mass of cloud, which was probably the cirrostratus which had been seen from the earth. About this time, and while still rising, a few small star-shaped crystals of snow about $\frac{1}{35}$th of an inch diameter fell upon us. The sun was almost constantly obscured throughout the ascent. The descent commenced at $4^h\ 46^m$ P.M., and the earth was reached about $5^h\ 20^m$ P.M., near Swavesey in Cambridgeshire, about 57 miles north of London. There seems to have been little, if any variation in the direction of the balloon's flight: it would

thus appear that, within the height reached by us, the air was moving from south, at an average rate of about 38 miles an hour. A violent thunder-storm, with heavy rain, occurred about two hours after the descent took place, some symptoms of which were at one time noticed from the balloon at a great distance.

In this ascent it was found impossible to use the aspirator, which was too large when two persons were seated. The free dry and wet thermometers were regularly observed. Some specimens of air were collected during the descent, and supplied to Dr. MILLER.

Second Ascent, August 26.—The wind on the 25th blew strongly from the west, but lulled in the evening. On the 26th it blew from east with moderate force; the sky was to a small extent obscured by detached masses of cumulus, and the sun shone brightly. The ascent took place at $4^h\ 43^m$ P.M., and observations were commenced at $4^h\ 46^m$. The clouds were again passed without being perceived, their height, however, was estimated at 3000 feet: above this height no clouds were met with, the sky being exceedingly clear and of a very deep blue colour. The currents of air passed through seem to have been from various directions, but generally moving with little rapidity. On leaving Vauxhall the balloon was at first carried towards the west for about 2 miles: when it reached the height of 5000 or 6000 feet it began to move slowly towards N.N.E. for about 4 miles, until about $5^h\ 25^m$, at an elevation of 12,000 feet, the direction of its motion, which was still for some time very slow, became W.N.W.; this direction it seems to have maintained during the remainder of the ascent, and probably with increased rapidity. The descent commenced at $7^h\ 0^m$, and the balloon reached the earth at $7^h\ 35^m$ P.M. near Chesham in the county of Bucks, about 25 miles W.N.W. of London. On this occasion all the instruments were regularly observed: some difficulty was experienced in the observation of REGNAULT'S hygrometer, as the force of the aspirator was not sufficient to produce the great degree of cold required for the deposition of dew. This was remedied by Mr. NICKLIN, who, at the cost of some exertion, maintained an increased strain upon the aspirator during the observations. The sun shone brightly throughout the ascent. Specimens of air were again collected during the descent.

Third Ascent, October 21.—The weather had for a fortnight previously been fine, with an easterly wind; on the 19th the barometer began to fall and the east wind ceased; on the 20th the weather was fine, the air at the surface being calm, and the high clouds moving from S.W.: a fog existed on the night of the 20th, which slowly disappeared on the morning of the 21st, leaving the air in a very calm state and with some haze. A dense mass of cloud covered the sky, one or two slight showers falling about 10 A.M. I was the only observer on this occasion. The ascent commenced at $2^h\ 45^m$ P.M., and the balloon rose at first nearly vertically, but soon began to move towards E.N.E. Between the heights of 1000 and 2800 feet various detached and irregular masses of loose scud were encountered, but the balloon had not completely entered the dense mass of cloud till the height of nearly 3000 feet. At a

height of 3700 feet the upper surface of the cloud was reached, and the sun was seen shining through thin cirrous clouds, at a great height. The height of the upper surface of the cloud was again observed during the descent at $4^h\ 6^m$ to be 3450 feet. When the balloon was close to the clouds, it was remarked that the general level of the surface was very uniform, presenting, however, a hillocky appearance; the irregularities being small, apparently not exceeding a very few feet. Shortly after clearing the clouds, a shadow of the balloon was seen on the surface fringed with a glory; with this shadow as a centre, there was also observed a circle of whitish light, the outer edge of it slightly tinged with yellow; its diameter being estimated at 80°. About this time there was noticed, stretching for a considerable length in a serpentine course over the surface of the cloud, a well-defined belt having the appearance of a broad road, both sides being strikingly distinct. When the balloon had attained a height of above 12,000 feet, Mr. Green, who had been watching its motion with reference to the clouds below, decided that, as it seemed to be moving rapidly from N.W., it would be prudent to descend below the clouds, to ascertain our position with reference to the sea, and if there should be space enough to ascend a second time to a greater height. It was found, however, on descending, that we were already very near the sea, indeed, moving along the river Thames within a short distance of its mouth. A second ascent being thus unadvisable, the descent was made at $4^h\ 20^m$ P.M. on the North bank of the Thames, between the villages of South Benfleet and Rayleigh in Essex, about 30 miles east of London. The average rate of motion was thus about 18 miles an hour, but in the higher part of our course it must have been considerably more.

When about 3000 to 4000 feet above the clouds they were examined with the polariscope. The reflected light from the clouds *next* the sun showed no trace whatever of polarization: the slightly bluish-grey clouds on the side *from* the sun showed *very* slight symptoms of polarization, the light of the sky being strongly polarized.

Fourth Ascent, November 10.—This ascent had been delayed for some days, owing to the unfavourable state of the weather, the wind having been generally from a westerly quarter. On the 10th the surface wind and the lower current of scud were moving very slowly from about N.E.: the upper clouds were only occasionally visible, and seemed to proceed from about N.N.W. The ascent commenced at $2^h\ 21^m\ 40^s$ P.M. At a height of 500 feet the first cloud, thin scud, was entered, the *upper* surface being 1970 feet high. A space of 2000 feet was clear of clouds, and at 4000 feet the second stratum of clouds was reached, its *upper* surface being found to be 4900 feet high. After this no clouds were met with, the sun shining through thin cirrous clouds, which must have been at a very great height. From notes taken at Vauxhall by Mr. Gassiot, it appears that, at starting, the balloon moved towards south-west until $2^h\ 26^m$, when, just as it had reached the *upper* surface of the *first* stratum of clouds, or at a height of about 2000 feet, the direction became easterly. Bearings

and altitudes taken by Mr. GLAISHER show that at $2^h\ 44^m$, when the height was 11,000 feet, the balloon was 5 miles S. by E. of Greenwich Observatory. The greatest elevation (22,930 feet) was reached at $3^h\ 16^m$ P.M.; about which time the clouds, which had hitherto obscured the earth, had disappeared, and we perceived that the balloon was rapidly approaching the sea. Mr. GREEN discharged gas copiously, and the descent became very rapid; a landing being effected within 4 miles of the sea, accompanied by a considerable shock which broke several of the instruments. The descent took place, between $3^h\ 40^m$ and $3^h\ 45^m$, at Acryse near Folkstone, about 57 miles E.S.E. from London. The time occupied in moving from a little S.W. of Vauxhall to 5 miles S. by E. of Greenwich, or about 9 miles, was 18 minutes; the remainder of the distance to Acryse, about 50 miles, being accomplished in from 55 to 60 minutes, or at the rate of fully 50 miles an hour.

As the height reached on this occasion was considerably greater than in the previous ascents, the effect of the diminished pressure was more severely felt; both Mr. GREEN and myself having experienced considerable difficulty in respiration, with much breathlessness and fatigue after any muscular exertion.

§ 4. *Description of the Table of Observations.*

All the meteorological observations taken during the ascents are contained in Table I.

Column 1 contains the times at which the observations were made. Column 2 contains the readings of the thermometer attached to the barometer. Column 3 contains the observations of the barometer corrected for temperature, by SCHUMACHER's tables, and for scale error. The numbers, to which the mark † is affixed, in the observations of August 17 and 26, are the occasional readings by myself of the second barometer. The readings of the barometer were made by estimation to 0·01 inch; but the probable error of an observation, from various causes,—such as rapid change in the height, and the occasional oscillation of the mercury from agitation of the car,—is perhaps 0·03 inch, or even sometimes more. This degree of accuracy appears, however, to be quite sufficient with reference to the changes of the temperature and humidity; an error of 30 or 40 feet in the resulting height being equivalent in general to a change of only one-tenth of a degree of temperature.

Column 4 contains the height above the level of the sea, as deduced from the barometric readings by the formula of LAPLACE. The formula actually employed was

$$z = \log\left(\frac{h}{h'}\right) \times 18336\left(1+\frac{2(t+t')}{1000}\right)\left(1+0{\cdot}002837\cos 2L\right)\left(1+\frac{z+15926}{6366200}\right)^{*};$$

or expressed in English feet and FAHRENHEIT's degrees,

$$z = \log\left(\frac{h}{h'}\right) \times 60159\left(1+\frac{t+t'-64}{900}\right)\left(1+0{\cdot}002837\cos 2L\right)\left(1+\frac{z+52251}{20886900}\right),$$

where z is the height required; h and h', t and t' the height of the barometer cor-

* Annuaire Météorologique de la France, 1849, page 54.

rected for temperature, and the temperature of the air, at the lower and upper stations respectively; L, the latitude. The temperature of the air for the position of the balloon has been derived from the readings of the aspirated dry thermometer (column 5), except on August 17, when the free thermometer only was observed. The temperature and barometric height at the earth's surface have been taken by interpolation from the comparative observations at different stations; the mean height above the sea, of the stations referred to, having been included. The numbers, it will be seen, have been given only to the nearest 10 feet.

Many observers in different parts of the country made corresponding meteorological observations, generally at hourly intervals, on the days of the several ascents. These have been arranged in compact tabular order by Colonel Sykes, Chairman of the Kew Committee, and are appended to this report. The stations selected for comparison with the different days' observations have been those which lay nearest to the course of the balloon. The temperature of the air at the surface of the earth, has been derived from the mean of the observations at all the selected stations, both as regards its *absolute value* and *hourly change*. The *hourly change* of the barometer has been taken from the observations at all the selected stations; but its *absolute height* has always been derived from the mean of the observations at the Royal Observatory, Greenwich, and at the residence of James Glaisher, Esq., Lewisham. The error likely to result from adopting the height at these two stations as the standard of reference will be in any case very small, and can only affect the *absolute* and not the *relative* heights of the balloon by a few feet; while any uncertainty with regard to the index errors of other barometers is obviated. The quantity +0·028 has been added to the readings of the terrestrial barometers, on account of the index errors of the balloon barometers.

The following are the stations whose observations have been employed, and the resulting mean values for each day of ascent.

August 17, 5 stations, viz.—Greenwich; Lewisham; Enfield; St. John's Wood; Cambridge.

Mean temperature of the air at 4^h P.M. $=71°{\cdot}2$; hourly change $=-1°{\cdot}1$.

Mean height of the barometer at 4^h P.M. $=29{\cdot}740$ in.; hourly change $=-0{\cdot}036$ in.

August 26, 5 stations, viz.:—Greenwich; Lewisham; St. John's Wood; Kew Observatory; Stone Rectory, Bucks.

	Temperature of the Air.		Barometer.	
Time.	Mean.	Hourly Change.	Mean Height.	Hourly Change.
h	°	°	in.	in.
4 P.M.	69·7		29·949	
		−2·2		+0·010
5	67·5		·959	
		−2·3		+0·005
6	65·2		·964	
		−2·5		+0·021
7	62·7		·985	

October 21, 2 stations, viz.—Greenwich and Lewisham.

Time. h m	Temperature of the Air. Mean.	Half-hourly Change.	Barometer. Mean Height. in.	Half-hourly Change. in.
2 30 P.M.	58·7		29·900	
		0·0		—0·001
3 0	58·7		·899	
		—1·0		— ·004
3 30	57·7		·895	
		—0·8		— ·007
4 0	56·9		·888	
		—0·8		·000
4 30	56·1		·888	

November 10, 2 stations,—Greenwich and Lewisham.

Time. h m	Temperature of the Air. Greenwich.	Lewisham.	Mean.	Barometer. Mean Height. in.	Half-hourly Change. in.
2 30 P.M.	48·6	50·7	49·7	29·978	
					—0·003
3 0	48·0	49·3	48·7	·975	
					— ·004
3 30	49·0	49·6	49·3	·971	
	Mean		49·2		

As the progress of the temperature at these two stations has been very irregular and indefinite, a mean result has been adopted, and no allowance made for hourly change.

The height, above the mean sea level, of Greenwich =159 feet.

The height, above the mean sea level, of Lewisham = 80 feet.

Mean of both stations . . . . . =120 feet.

Columns 5–10 contain the results of observations with the aspirated dry and wet thermometers; the tension of vapour, relative humidity (100 being complete saturation), and the calculated temperature of the dew-point having been deduced by Dr. Apjohn's formula and Dalton's Tables of the elasticity of vapour. Column 11 contains the readings of the dry thermometer, corrected for hourly change by means of the numbers deduced above from observations at different stations. The numbers in this column have been employed in the subsequent discussions and in the projected results.

Columns 12–17 contain the observations of the free dry and wet thermometers similarly reduced. Columns 18 and 19 contain the results of the direct dew-point observations with Regnault's Hygrometer, and the corresponding tension of vapour derived from Dalton's Table. When numbers are entered in column 18 with the sign — after them, it is meant that the temperature in the hygrometer had been lowered to the degree stated, but that no dew was deposited.

All the readings of both pairs of dry and wet thermometers have been corrected for index error; the corrections to the dew-point thermometer were very small, and have been neglected.

TABLE I.—Meteorological Observations made in the Four Balloon Ascents of August 17, August 26, October 21, and November 10, 1852.

Greenwich Mean Time.			Barometer.		Height above sea-level.	Dry and Wet Thermometers, *aspirated*.					Temperature corrected for change.	Dry and Wet Thermometers, *free*.					Regnault's Hygrometer.	
			Therm.	Reading corrected.		Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.		Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Dew-point.	Tension of vapour.
	h	m	s	°	in.	feet.	°	°	°	in.		°	°		°	in.		°	°	in.
August 17, 1852.	3	52	0	.....	27·40	2,440						62·8	63·0	60·4	2·6	·503	87	58·8		
		54	0	.....	26·40	3,470						59·2	59·4	57·1	2·3	·451	88	55·6		
		54	0	70	26·42†	3,450														
		55	0	.....	25·80	4,110						58·1	58·2	55·3	2·9	·418	85	53·3		
		59	0	.....	24·22	5,880						57·8	57·8	43·4	14·4	·164	34	26·6		
		59	18	66·5	24·10†	6,020														
		59	30	.....	23·62	6,580														
	4	0	30	.....								54·1	54·1	40·5	13·6	·147	34	23·5		
		1	0	.....	23·42	6,800						54·1	54·1	40·8	13·3	·153	36	24·7		
		1	30	.....	23·33	6,910						53·7	53·7	40·1	13·6	·144	34	23·0		
		2	0	.....	23·13	7,140						53·1	53·1	39·8	13·3	·145	35	23·2		
		2	0	63·0	22·96†	7,350														
		2	30	.....	22·93	7,380						52·3	52·3	39·3	13·0	·144	36	23·0		
		3	0	.....	22·78	7,550						51·4	51·3	38·4	12·9	·138	35	21·8		
		3	30	.....	22·83	7,480						50·5	50·4	37·7	12·7	·133	35	20·8		
		4	0	.....	22·43	7,970						49·8	49·7	37·4	12·3	·136	37	21·4		
		4	30	.....	22·33	8,100						49·8	49·7	38·3	11·4	·152	41	24·5		
		5	0	.....	22·14	8,340						50·5	50·4	39·2	11·2	·162	43	26·2		
		5	0	61·0	22·13†	8,350														
		5	30	.....	22·04	8,460						50·2	50·1	39·6	10·5	·172	46	27·9		
		6	0	.....	21·94	8,590						49·8	49·7	39·1	10·6	·167	45	27·1		
		6	30	.....	21·84	8,710						49·2	49·2	38·3	10·9	·158	44	25·5		
		7	0	.....	21·84	8,700						48·0	47·9	37·3	10·6	·152	44	24·5		
		7	30	.....	21·84	8,690						47·3	47·2	39·4	7·8	·194	57	31·3		
		8	0	.....	21·74	8,810						47·5	47·3	40·5	6·8	·212	63	33·8		
		9	30	.....	21·49	9,130						47·0	46·8	37·1	9·7	·159	48	25·7		
		10	0	.....	21·49	9,120						45·9	45·7	36·7	9·0	·161	50	26·1		
		10	30	.....	21·34	9,310						44·9	44·7	34·5	10·2	·135	43	21·2		
		11	0	.....	21·24	9,430						44·0	43·8	33·2	10·6	·122	40	18·4		
		11	30	.....	21·13	9,560						43·3	43·1	32·5	10·6	·117	40	17·3		
		12	0	.....	21·00	9,730						42·8	42·6	32·3	10·3	·119	41	17·8		
		12	35	57·0	20·78†	10,020						43·1	42·8	32·5	10·3	·121	42	18·2		
		13	0	.....	20·65	10,180						41·8	41·5	31·7	9·8	·127	46	19·5		
		13	0	.....	20·62†	10,220														
		13	30	.....	20·45	10,440														
		14	0	.....	20·25	10,700						40·4	40·1	31·2	8·9	·132	50	20·6		
		14	0	54·0	20·27†	10,680														
		14	30	.....	20·10	10,900						40·0	39·7	30·3	9·4	·122	47	18·4		
		15	0	.....	20·06	10,950						39·4	39·1	31·9	7·2	·149	58	23·9		
		15	15	.....	19·76†	11,360														
		15	30	.....	19·80	11,290						38·1	37·8	32·0	5·8	·160	65	25·9		
		16	0	.....	19·52	11,680						37·3	37·0	31·7	5·3	·162	68	26·2		
		16	15	.....	19·57†	11,600														
		16	30	.....	19·51	11,690						37·0	36·7	31·5	5·2	·161	68	26·1		
		17	0	.....	19·40	11,840						36·5	36·2	31·2	5·0	·160	69	25·9		
		17	30	.....	19·35	11,900														
		18	0	.....	19·25	12,040						36·1	35·7	30·8	4·9	·158	69	25·5		
		18	30	.....	19·15	12,180						35·7	35·3	30·8	4·5	·161	70	26·1		
		19	0	.....	19·14	12,200														
		19	30	.....	19·15	12,190						36·5	36·1	32·9	3·2	·182	79	29·5		
		20	0	.....	19·16	12,150						34·8	34·4	31·4	3·0	·175	81	28·4		
		20	20	48·0	19·15†	12,160														
		20	30	.....	19·16	12,140						34·1	33·7	29·9	3·8	·160	76	25·9		
		21	0	.....	19·15	12,160						34·5	34·1	30·1	4·0	·160	75	25·9		
		21	30	.....	19·06	12,280						33·7	33·3	31·5	1·8	·184	88	29·8		
		22	0	.....	18·91	12,500														
		22	20	47·0	18·87†	12,550														
		22	30	.....	18·66	12,860						33·9	33·4	31·4	2·0	·182	87	29·5		

Table I. (Continued.)

Greenwich Mean Time.			Barometer.		Height above sea-level.	Dry and Wet Thermometers, *aspirated*.					Temperature corrected for change.	Dry and Wet Thermometers, *free*.					Regnault's Hygrometer.	
			Therm.	Reading corrected.		Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.		Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Dew-point.	Tension of vapour.
	h	m	s	in.	feet.	°	°	°	in.		°	°	°	°	°	in.		°	°	in.
August 17, 1852.	4	23	0	.....	18·56	13,000	.....	.....	.....	.....	.....	33·7	33·2	31·3	1·9	·182	88	29·5		
		24	0	.....	18·21	13,470	.....	.....	.....	.....	.....	30·3	29·8	28·4	1·4	·166	90	26·9		
		24	15	44·0	18·13†	13,590														
		24	30	.....	18·11	13,610														
		25	0	.....	17·87	13,960	.....	.....	.....	.....	.....	28·4	27·9	26·6	1·3	·156	89	25·2		
		25	30	.....	17·77	14,100	.....	.....	.....	.....	.....	27·5	27·0	25·9	1·1	·153	92	24·7		
		25	30	.....	17·82†	14,020														
		26	0	.....	17·47	14,550	.....	.....	.....	.....	.....	27·0	26·5	25·0	1·5	·146	89	23·4		
		26	20	.....	17·39†	14,680														
		26	30	.....	17·37	14,710	.....	.....	.....	.....	.....	27·1	26·6	24·6	2·0	·141	86	22·4		
		27	0	.....	17·27	14,850	.....	.....	.....	.....	.....	26·3	25·8	24·2	1·6	·141	88	22·4		
		27	15	39·0	17·08†	15,150														
		27	30	.....	17·06	15,170	.....	.....	.....	.....	.....	25·5	24·9	23·4	1·5	·137	89	21·6		
		27	45	38·0	16·97†	15,310														
		28	0	.....	16·97	15,310	.....	.....	.....	.....	.....	24·9	24·3	23·0	1·3	·136	90	21·4		
		28	20	.....	16·98†	15,290														
		28	30	.....	16·96	15,330	.....	.....	.....	.....	.....	25·3	24·7	23·0	1·7	·134	87	21·0		
		29	0	.....	16·86	15,480	.....	.....	.....	.....	.....	24·9	24·3	22·6	1·7	·132	87	20·6		
		29	30	.....	16·67	15,780	.....	.....	.....	.....	.....	24·6	24·0	(27·)						
		30	0	.....	16·67	15,780														
		31	0	37·0	16·52†	15,970	.....	.....	.....	.....	.....	21·5	20·9							
		31	30	.....	16·37†	16,200	.....	.....	.....	.....	.....	20·4	19·8	18·4	1·4	·114	89	16·6		
		32	0	35·0	16·32†	16,280	.....	.....	.....	.....	.....	20·0	19·4	18·0	1·4	·112	88	16·1		
		33	0	.....	16·26	16,380	.....	.....	.....	.....	.....	20·5	19·8	18·3	1·5	·113	88	16·4		
		33	30	.....	16·16	16,550	.....	.....	.....	.....	.....	20·5	19·8	17·7	2·1	·107	84	14·9		
		34	0	.....	16·16	16,560	.....	.....	.....	.....	.....	21·6	20·9	18·8	2·1	·112	84	16·1		
		34	15	.....	16·06†	16,730														
		34	30	.....	16·11	16,650	.....	.....	.....	.....	.....	21·7	21·0	16·1	4·9	·085	63	8·7		
		35	0	.....	16·06	16,730														
		35	30	.....	16·06	16,720	.....	.....	.....	.....	.....	21·0	20·3							
		36	0	32·0	15·96	16,880	.....	.....	.....	.....	.....	20·4	19·7							
		36	30	.....	15·96	16,870	.....	.....	.....	.....	.....	20·0	19·3							
		37	0	.....	15·96	16,870	.....	.....	.....	.....	.....	19·6	18·9							
		37	0	.....	15·92†	16,930														
		37	30	.....	15·87	17,020	.....	.....	.....	.....	.....	20·0	19·3							
		38	0	.....	15·86	17,050	.....	.....	.....	.....	.....	20·4	19·7	17·7	2·0	·108	84	15·1		
		38	0	.....	15·80†	17,150														
		38	30	.....	15·67	17,370	.....	.....	.....	.....	.....	20·5	19·8	17·0	2·8	·101	79	13·3		
		39	0	.....	15·57	17,530	.....	.....	.....	.....	.....	19·5	18·8	16·0	2·8	·096	78	12·0		
		39	30	.....	15·47	17,680	.....	.....	.....	.....	.....	18·8	18·1	14·1	4·0	·083	69	8·1		
		40	0	30·0	15·37	17,860	.....	.....	.....	.....	.....	18·6	17·9	14·4	3·5	·087	73	9·3		
		40	0	.. ..	15·32†	17,950														
		40	30	.....	15·17	18,200	.....	.....	.....	.....	.....	17·9	17·2							
		41	0	.....	15·17	18,170	.....	.....	.....	.....	.....	16·4	15·7	9·4	6·3	·054	49	−3·2		
		41	30	.. ..	15·07	18,310	.....	.....	.....	.....	.....	14·5	13·8	7·7	6·1	·050	49	−5·2		
		42	0	.....	14·86	18,670	.....	.....	.....	.....	.....	14·0	13·3	7·2	6·1	·049	49	−5·8		
		42	15	.....	14·67†	19,010														
		42	30	.....	14·81	18,750	.....	.....	.....	.....	.....	13·6	12·8	7·3	5·5	·053	53	−3·7		
		43	0	.....	14·76	18,840	.....	.....	.....	.....	.....	13·5	12·7	8·1	4·6	·060	61	−0·5		
		43	15	28·0	14·62†	19,090														
		43	30	.....	14·66	19,020	.....	.....	.....	.....	.....	13·2	12·4	7·5	4·9	·057	58	−1·8		
		43	45	.....	14·54†	19,220														
		44	0	.....	14·56	19,189	.....	.....	.....	.....	.....	12·1	11·3	7·2	4·1	·060	64	−0·5		
		44	30	.....	14·46	19,340	.....	.....	.....	.....	.....	11·2	10·4	6·7	3·7	·061	67	−0·1		
		45	0	.....	14·46	19,310	.....	.....	.....	.....	.....	9·8	9·0	6·4	2·6	·065	76	+1·6		
		45	0	.....	14·36†	19,500														
		45	30	.....	14·41	19,420	.....	.....	.....	.....	.....	10·5	9·7	7·7	2·0	·072	82	4·3		
		46	0	.....	14·35	19,510	.....	.....	.....	.....	.....	9·5	8·7	6·6	2·1	·068	80	2·8		
		46	30	.....	14·41†	19,380	.....	.....	.....	.....	.....	8·7	7·9	6·3	1·6	·070	85	3·6		

Table I. (Continued.)

Greenwich Mean Time.			Barometer.		Height above sea-level.	Dry and Wet Thermometers, *aspirated*.					Temperature corrected for change.	Dry and Wet Thermometers, *free*.					Regnault's Hygrometer.	
			Therm.	Reading corrected.		Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.		Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Dew-point.	Tension of vapour.
	h	m	s	°	in.	feet.	°	°	°	in.		°	°	°	°	in.		°	°	in.
August 17, 1852.	4	47	0	.....	.....	.....						8·9	8·1							
		47	30	.....	14·46	19,300														
		49	0	.....	14·86	18,630						12·3	11·4							
		50	0	.....	15·17	18,090						13·3	12·4							
		50	30	.....	15·27	17,920						13·3	12·4							
		51	0	.....	15·48	17,580						14·6	13·7							
		51	30	24·0	15·58	17,420						15·4	14·5							
		51	30	.....	15·63†	17,330														
		52	0	.....	15·88	16,920						15·9	15·0							
		52	15	24·0	15·83†	17,010														
		52	30	.....	16·08	16,590														
		53	0	.....	16·39	16,090						17·2	16·2	15·2	1·0	·103	92	13·9		
		54	0	.....	16·59	15,790						19·0	18·0	16·2	1·8	·102	85	13·6		
		54	30	.....	16·89	15.320						19·5	18·5							
		55	0	.....	17·10	15,000						20·5	19·5	18·8	0·7	·120	94	18·0		
		56	45	.....	17·77†	14,020														
		57	0	.....	.....	.....						25·0	24·0							
		57	40	.....	18·11†	13,530						26·0	25·0							
		58	0	24·0	18·36†	13,170						27·3	26·2							
		58	30	.....	18·71†	12,680						28·4	27·3							
	5	0	30	.....	19·57†	11,500						32·0	30·9	29·8	1·1	·177	92	28·7		
		1	30	.....	20·28†	10,570						36·5	35·4	29·8	5·6	·145	64	23·2		
		3	0	.....	20·88†	9,790						38·0	36·9	29·8	7·1	·133	56	20·8		
		5	0	.....	22·31	8,020						43·0	41·8	30·8	11·0	·106	38	14·6		
		5	30	.....	22·51	7,780						45·4	44·2	31·5	12·7	·096	31	12·0		
		6	0	.....	22·91	7,310						48·0	46·8							
August 26, 1852.	4	46	0	.....	.....	.....	65·9	61·5	4·4	·500	79	58·7	65·9	66·0	61·7	4·3	·505	79	59·0		
		47	0	.....	29·34	710	64·5	60·9	3·6	·498	82	58·5	64·5	65·3	61·3	4·0	·501	80	58·7		
		47	30	71·0	29·18†	860														
		48	0	.....	29·19	850	63·8	59·9	3·9	·478	80	57·3	63·9	64·8	60·7	4·1	·490	80	58·1		
		49	0	.....	28·89	1,140	63·2	59·5	3·7	·473	81	57·0	63·3	63·8	59·7	4·1	·472	79	56·9		
		50	0	.....	28·40	1,620	61·1	57·8	3·3	·450	83	55·5	61·2	62·5	58·1	4·4	·443	78	55·1		
		51	0	.....	28·10	1,920															
		51	40	.....	.....	.....						.....	.....	.....	.....	.....	.....	.....	.....	50·0	·373
		52	0	.....	27·85	2,160						.....	60·9	57·3	3·6	·440	82	54·9		
		52	30	.....	.....	.....	59·9	56·2	3·7	·421	80	53·5	60·1								
		53	0	.....	27·70	2,310	58·9	55·8	3·1	·421	83	53·5	59·1	60·9	56·9	4·0	·429	79	54·1		
		53	30	.....	.....	.....	58·9	55·5	3·4	·414	82	53·0	59·2								
		54	0	.....	27·40	2,620	58·3	55·3	3·0	·415	84	53·1	58·6	60·3	55·7	4·6	·405	77	52·4		
		54	30	.....	.....	.....	58·0	54·8	3·2	·406	83	52·5	58·3	59·6	55·6	4·0	·410	80	52·8		
		55	0	.....	27·11	2,910															
		55	30	.....	.....	.....	56·3	54·6	1·7	·419	91	53·4	56·7								
		56	0	.....	26·90	3,120	55·8	53·2	2·6	·389	86	51·2	56·2	57·7	53·7	4·0	·382	79	50·7		
		57	0	.....	26·41	3,630	54·8	52·0	2·8	·371	84	49·8	55·2	56·8	52·9	3·9	·372	79	49·9		
		57	30	.....	.....	.....	54·4	51·6	2·8	·366	84	49·4	54·8								
		58	0	.....	26·16	3,900															
		58	0	65·3	26·17†	3,890															
		58	30	.....	.....	.....	52·5	50·0	2·5	·348	86	48·0	52·9	54·9	50·7	4·2	·341	77	47·4		
		59	0	.....	25·77	4,300	51·5	48·7	2·8	·329	84	46·3	52·0	54·1	49·6	4·5	·324	76	45·9	44·3	·307
		59	30	.....	.....	.....	51·0	47·9	3·1	·317	82	45·2	51·5								
	5	0	0	.....	25·32	4,780	49·8	46·9	2·9	·307	83	44·3	50·3	52·4	48·6	3·8	·319	79	45·4	40·5	·269
		1	0	.....	24·92	5,230	49·8	46·6	3·2	·302	82	43·8	50·3	50·5	46·8	3·7	·299	79	43·6		
		2	0	.....	24·47	5,710	46·7	45·4	1·3	·307	92	44·3	47·3								
		2	30	.....	.....	.....	45·7	44·3	1·4	·294	91	43·1	46·3							41·?	·274
		3	0	.....	24·12	6,100	44·7	43·0	1·7	·278	89	41·5	45·3	47·1	44·2	2·9	·279	83	41·6		
		3	30	.....	.....	.....	43·8	42·4	1·4	·274	91	41·0	44·4	46·1	43·3	2·8	·271	83	40·7	39·0	·255
		4	0	.....	23·68	6,590	42·7	41·1	1·6	·260	89	39·5	43·3	44·9	42·7	2·2	·270	86	40·6		
		4	40	.....	.....	.....	41·8	40·0	1·8	·248	88	38·2	42·5	.....	.....	.....	.....	.....	.....	35·5	·226

Table I. (Continued.)

Greenwich Mean Time.	Barometer. Therm.	Barometer. Reading corrected.	Height above sea-level.	Dry and Wet Thermometers, *aspirated*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Temperature corrected for change.	Dry and Wet Thermometers, *free*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Regnault's Hygrometer. Dew-point.	Tension of vapour.
h m s	°	in.	feet.	°	°	°	in.		°	°	°	°	°	in.		°	°	in.
August 26, 1852. 5 5 0	.....	23·13	7,230	41·0	39·8	1·2	·252	92	38·7	41·7	43·6	40·9	2·7	·249	83	38·3		
6 0	.....	22·78	7,650															
6 30	.....		.....	40·8	35·7	5·1	·183	67	29·6	41·5								
7 0	.....	22·33	8,200															
8 0	.....	22·08	8,510	42·3	31·0	11·3	·107	37	14·9	43·1								
8 30	.....		.....	42·4	30·0	12·4	·092	32	10·8	43·2							14·—	
9 0	.....	21·74	8,940	41·7	29·0	12·7	·073	26	4·7	42·5	42·7	32·0	10·7	·110	38	15·7		
9 30	.....		.....	41·1	28·5	12·6	·082	30	7·8	42·0							10·—	
10 0	.....	21·54	9,190	41·3	29·0	12·3	·088	32	9·6	42·2	41·7	31·7	10·0	·123	44	18·7	7·—	
10 30	.....		.....	41·7	30·0	11·7	·099	35	12·8	42·6							6·0	·077
11 0	.....	21·44	9,320								43·6	34·9	8·7	·150	50	24·1		
11 30	.....		.....	42·5	32·0	10·5	·113	39	16·4	43·4	43·6	33·0	10·6	·120	40	18·0		
12 0	.....	21·32	9,480															
12 30	.....		.....	41·8	29·5	12·3	·092	33	10·8	42·8	44·0	32·0	12·0	·102	34	13·6		
13 0	.....	21·24	9,580	42·3	29·5	12·8	·088	31	9·6	43·3								
13 30	58·4		.....	40·9	27·2	13·7	·068	25	2·8	41·9								
14 0	.....	21·14	9,690	39·8	26·6	13·2	·069	26	3·2	40·8	42·7	30·1	12·6	·095	33	11·7		
15 0	.....	21·04	9,820	39·9	27·1	12·8	·075	28	5·4	41·0	43·6	31·2	12·4	·104	35	14·1		
16 0	.....	20·94	9,970	42·0	30·6	11·4	·107	38	14·9	43·1	44·5	33·4	11·1	·120	39	18·0		
16 30	.....		.....	42·4	31·2	11·2	·113	39	16·4	43·5								
17 0	.....	20·75	10,200	39·8	28·5	11·3	·095	36	11·7	40·9	42·3	31·6	10·7	·120	42	18·0		
17 0	.....	20·74†	10,210															
18 0	.....	20·70	10,260	40·0	29·7	10·3	·108	41	15·1	41·2	42·3	32·0	10·3	·118	41	17·5		
18 30	.....		.....	41·8	31·0	10·8	·113	40	16·4	43·0								
19 0	.....	20·70	10,290	42·7	32·0	10·7	·114	39	16·6	43·9	43·4	34·6	8·8	·149	50	23·9		
20 0	.....	20·85	10,100	43·8	32·6	11·2	·114	38	16·6	45·1								
21 0	.....	20·75	10,200	40·5	28·8	11·7	·094	35	11·4	41·8								
22 0	58·0	20·65	10,340	40·8	29·5	11·3	·102	38	13·6	42·1							12·—	
22 0		20·59†	10,420														10·8	·092
22 30	.....		.....	40·8	29·4	11·4	·101	37	13·4	42·1								
23 0	.....	20·43	10,620	40·2	28·4	11·8	·092	35	10·8	41·6	42·8	32·4	10·4	·121	41	18·2		
23 30	.....		.....	39·5	28·1	11·4	·094	36	11·4	40·9								
24 0	.....	20·25·	10,840	37·6	27·3	10·3	·099	41	12·8	39·0							6·0	·077
24 30	.....		.....	37·2	27·1	10·1	·097	40	12·2	38·6							6·0	·077
24 40	.....		.....								38·9	29·4	9·5	·116	46	17·1		
25 0	.....	20·10	11,050	38·5	28·1	10·4	·102	41	13·6	40·0	39·4	30·5	8·9	·127	49	19·5		
26 0	.....	20·15	10,980	38·4	28·1	10·3	·102	41	13·6	39·9	39·0	30·2	8·8	·126	49	19·3		
26 30	.....		.....	38·8	29·1	9·7	·113	45	16·4	40·3								
27 0	.....	20·20	10,920	39·4	29·1	10·3	·109	42	15·4	40·9								
27 30	.....		.....	39·1	28·5	10·6	·102	40	13·6	40·6	39·4	30·9	8·5	·133	52	20·8		
28 0	.....	20·28	10,810	38·0	26·6	11·4	·085	34	8·7	39·6								
28 30	.....		.....	36·8	26·6	10·2	·093	39	11·1	38·4								
29 0	.....	20·14	10,970	35·4	25·9	9·5	·094	42	11·4	37·0							10·—	
29 30	.....		.....	34·0	25·1	8·9	·094	44	11·4	35·6	36·2	27·1	9·1	·104	45	14·1		
30 0	.....	19·95	11,210	33·9	25·0	8·9	·094	44	11·4	35·5							8·0	·083
30 30	.....		.....	36·0	26·9	9·1	·104	45	14·1	37·7								
31 0	.....	19·80	11,440	35·9	27·0	8·9	·106	46	14·6	37·6	35·6	28·1	7·5	·122	54	18·4		
31 30	.....		.....	35·4	26·4	9·0	·102	45	13·6	37·1	36·7	29·4	7·3	·132	56	20·6		
32 0	.....	19·73	11,540	36·8	27·2	9·6	·103	44	13·9	38·5								
32 30	.....		.....	37·5	27·6	9·9	·103	43	13·9	39·2	39·5	32·0	7·5	·142	55	22·6		
33 0	.....	19·65	11,650	37·2	26·8	10·4	·095	40	11·7	39·0								
34 0	.....	19·59	11,740	37·3	27·2	10·1	·100	42	13·1	39·1	40·0	31·6	8·4	·140	53	22·2		
35 0	.....	19·59	11,740	38·0	27·7	10·3	·102	41	13·6	39·8								
35 30	.....		.....	38·8	28·1	10·7	·101	40	13·3	40·6	38·0	30·0	8·0	·131	53	20·4	12·0	·096
36 0	.....	19·60	11,730															
36 50	.....		.....	36·8	27·2	9·6	·103	44	13·9	38·7								
37 0	.....	19·64	11,650								40·7	33·3	7·4	·153	56	24·7		
37 30	.....		.....	35·7	25·8	9·9	·093	41	11·1	37·6								
38 0	.....	19·60	11,690	35·0	25·1	9·9	·089	40	9·9	36·9								

TABLE I. (Continued.)

Greenwich Mean Time.			Barometer.		Height above sea-level.	Dry and Wet Thermometers, *aspirated*.						Temperature corrected for change.	Dry and Wet Thermometers, *free*.						Regnault's Hygrometer.	
			Therm.	Reading corrected.		Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.		Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Dew-point.	Tension of vapour.
	h	m	s	°	in.	feet	°	°	°	in.		°	°	°	°	in.		°	°	in.
August 26, 1852.	5	38	30	.....	.....	.....	33·6	23·8	9·8	·082	39	7·7	35·6	°	°	°	in.		°	°	in.
		39	0	.....	19·55	11,730	32·5	22·8	9·7	·077	38	6·1	34·5								
		39	30	.....	.....	.....	32·9	22·8	10·1	·075	36	5·4	34·9								
		40	0	54·7	19·52	11,790	34·0	23·3	10·7	·074	35	5·0	36·0	.....	.....	.....	.....	.....	6·0	·077
		40	30	.....	19·50†	11,840	35·8	25·6	10·2	·090	40	10·2	37·8							
		41	0	.....	19·50	11,830	34·7	23·6	11·1	·073	33	4·7	36·8							
		41	30	.....	.....	.....	33·9	23·2	10·7	·074	35	5·0	36·0							
		42	0	.....	19·32	12,050	32·5	22·3	10·2	·073	36	4·7	34·6	37·5	28·9	8·6	·121	50	18·2	
		42	30	51·5	19·27†	12,150	34·9	24·7	10·2	·085	38	8·7	37·0							
		43	0	.....	19·20	12,220	32·7	22·3	10·4	·072	35	4·3	34·8	36·8	28·1	8·7	·116	49	17·1	
		43	30	.....	.....	.....	31·9	21·9	10·0	·073	37	4·7	34·1	.....	.....	.....	.....	.....	.....	0·—	
		44	0	52·5	19·15	12,280	31·9	22·1	9·8	·075	38	5·4	34·1							—3·—	
		44	30	.....	.....	.....	31·2	21·2	10·0	·069	36	3·2	33·4	33·3	.....	.....	.....	.....	.....	—5·0	·050
		45	0	52·5	19·10	12,330															
		46	0	52·5	19·07	12,360	29·0	21·3	7·7	·085	47	8·7	31·3								
		46	30	.....	.....	.....	29·2	21·1	8·1	·081	45	7·4	31·5								
		47	0	.....	19·05	12,390	29·0	21·1	7·9	·083	46	8·1	31·3								
		47	30	.....	.....	.....	30·9	23·2	7·7	·091	47	10·5	33·2	.....	.....	.....	.....	.....	.....	+4·0	·071
		48	0	53·6	19·05	12,420	31·5	23·4	8·1	·093	47	11·1	33·8								
		48	30	.....	.....	.....	32·5	24·2	8·3	·096	47	12·0	34·8								
		49	0	.....	19·04	12,470	34·7	26·2	8·5	·106	48	14·6	37·1								
		49	30	.....	.....	.....	34·1	26·1	8·0	·109	51	15·4	36·5								
		50	0	.....	19·00	12,480	30·6	23·9	6·7	·105	55	14·4	33·0								
		50	30	.....	.....	.....	30·9	24·7	6·2	·113	59	16·4	33·3							0·0	·061
		51	0	54·5	18·90	12,610	29·9	.....	.....	.....	.....	.....	32·3	32·7	25·2	7·5	·107	53	14·9		
		51	30	.....	.....	.....	29·6	21·9	7·7	·088	48	9·6	32·1								
		52	0	.....	18·80	12,740	28·9	21·3	7·6	·086	48	9·0	31·4								
		52	30	.....	.....	.....	27·6	20·5	7·1	·086	50	9·0	30·1								
		53	0	54·6	18·71	12,840	27·0	20·3	6·7	·087	52	9·3	29·5								
		53	30	.....	.....	.....	27·2	20·4	6·8	·087	52	9·3	29·7	.....	.....	.....	.....	.....	.....	—4·0	·052
		54	0	54·8	18·70	12,890	29·5	22·5	7·0	·096	53	12·0	32·1								
		55	0	.....	.....	.....	28·7	21·8	6·9	·093	52	11·1	31·3								
		56	0	.....	18·68	12,930	30·0	23·2	6·8	·101	54	13·3	32·6								
		56	30	.....	18·87†	12,690															
		57	0	55·4	18·78	12,860	35·4	27·1	8·3	·114	51	16·6	38·1								
		57	30	.....	.....	.....	35·1	27·1	8·0	·115	52	16·8	37·8								
		58	0	56·5	18·94	12,610	34·2	25·9	8·3	·106	49	14·6	36·9								
		58	30	.....	.....	.....	34·9	26·4	8·5	·107	49	14·9	37·6								
		59	0	.....	19·20	12,260	35·8	27·3	8·5	·113	50	16·4	38·6								
		59	30	.....	.....	.....	36·4	26·0	10·4	·091	39	10·5	39·2								
	6	0	0	.....	19·48	11,860	.....	.....	.....	.....	.....	.....	.....	35·7	25·2	10·5	·086	38	9·0		
		1	0	57·0	19·59	11,700	35·0	24·7	10·3	·084	38	8·4	37·8								
		2	0	58·2	19·59	11,690	34·6	24·9	9·7	·089	41	9·9	37·5								
		2	30	.....	.....	.....	32·5	24·2	8·3	·095	47	11·7	35·4								
		3	0	.....	19·45	11,870	33·1	25·2	7·9	·103	50	13·9	36·0								
		4	0	51·0	19·25	12,140	32·7	25·2	7·5	·106	50	14·6	35·7								
		4	0	.....	19·23†	12,170															
		5	0	.....	19·17†	12,220	29·5	23·2	6·3	·103	57	13·9	32·5								
		6	0	.....	19·17†	12,210	29·0	22·8	6·2	·102	57	13·6	32·0								
		7	30	.....	19·28†	12,110	33·5	26·8	6·7	·122	58	18·5	36·6								
		9	0	.....	.....	.....	33·9	27·0	6·9	·121	57	18·2	37·0	35·8	24·7	11·1	·079	35	6·8		
		11	0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	36·3	25·7	10·6	·088	38	9·6		
		12	0	.....	19·60	11,650															
		13	0	49·1	19·40	11,920	32·1	23·7	8·4	·091	45	10·5	35·4								
		13	30	.....	.....	.....	33·6	25·0	8·6	·098	46	12·5	37·0	34·7	24·5	10·2	·084	38	8·4		
		14	0	.....	19·26	12,130	32·4	24·4	8·0	·099	49	12·8	35·8	.....	.....	.....	.....	.....	.....	+5·—	
		15	0	47·8	19·16	12,270	32·9	25·2	7·7	·105	51	14·4	36·3	.....	.....	.....	.....	.....	.....	+2·5	·067
		15	50	.....	.....	.....	31·2	24·2	7·0	·105	54	14·4	34·7	.....	.....	.....	.....	.....	.....	+1·0	·064
		16	0	.....	19·01	12,460															

Table I. (Continued.)

Greenwich Mean Time.	Barometer. Therm.	Barometer. Reading corrected.	Height above sea-level.	Dry and Wet Thermometers, *aspirated*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Temperature corrected for change.	Dry and Wet Thermometers, *free*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Regnault's Hygrometer. Dew-point.	Tension of vapour.
h m s	°	in.	feet	°	°	°	in.		°	°	°	°	°	in.		°	°	in.
August 26, 1852. 6 16 10	.....	.....	.....	29·0	23·0	6·0	·105	59	14·4	32·5	35·3	23·7	11·6	·072	32	4·3		
16 30	.....	.....	.....	29·9	23·2	6·7	·102	55	13·6	33·4	32·2	22·5	9·7	·078	39	6·4		
17 0	47·3	18·95	12,530	29·7	23·2	6·5	·103	56	13·9	33·2	30·9	22·5	8·4	·087	45	9·3		
18 0	.....	18·85	12,670	29·0	22·7	6·3	·102	57	13·6	32·6	31·3	22·7	8·6	·087	45	9·3		
18 30	.....	.....	.....	30·0	23·3	6·7	·103	56	13·9	33·6	32·3	23·2	9·1	·087	43	9·3	− 1·0	·059
19 0	46·6	18·71	12,880	30·0	23·2	6·8	·101	55	13·8	33·6	32·1	22·7	9·4	·082	41	7·7	− 4·0	·052
19 40	.....	.....	.....	30·5	23·4	7·1	·101	53	13·3	34·1								
20 0	45·9	18·66	12,950	30·1	23·2	6·9	·101	54	13·3	33·7								
20 30	.....	.....	.....	29·9	23·2	6·7	·103	56	13·9	33·6	31·6	22·2	9·4	·080	41	7·1		
21 0	45·0	18·55	13,110															
21 40	44·6	18·49†	13,170															
22 0	.....	18·46	13,200	26·7	21·3	5·4	·101	61	13·3	30·4	29·0	20·5	8·5	·077	43	6·1		
23 0	44·5	18·26	13,500	27·4	21·9	5·5	·104	61	14·1	31·1	29·0	21·2	7·8	·086	48	9·0	+ 2·0	·066
24 0	.....	18·11	13,720	.....	.....	.....	.....	.....	.....	.....	29·5	21·9	7·6	·091	50	10·5		
25 0	43·0	17·97	13,920	26·7	20·1	6·6	·089	54	9·9	30·5	28·0	21·2	6·8	·093	54	11·1		
25 40	.....	.....	.....	26·0	19·4	6·6	·086	53	9·0	29·9	27·0	20·4	6·6	·090	54	10·2	0·0	·061
26 0	42·6	17·85	14,080	25·6	19·3	6·3	·087	55	9·3	29·5	26·4	20·5	5·9	·095	58	11·7		
26 30	.....	17·78†	14,220	27·7	20·5	7·2	·088	51	9·6	31·6	26·3	20·3	6·0	·094	58	11·4		
27 0	.....	17·77	14,180	24·2	18·5	5·7	·087	58	9·3	28·1	25·8	20·2	5·6	·096	60	12·0		
27 0	.....	17·75†	14,210															
28 0	.....	17·62	14,440	26·1	20·3	5·8	·095	59	11·7	30·1	26·1	20·4	5·7	·096	60	12·0		
28 30	.....	.....	.....	22·9	18·3	4·6	·094	65	11·4	26·9	25·1	19·6	5·5	·091	58	10·5		
29 0	.....	17·45	14,640	22·5	18·3	4·2	·087	62	9·3	26·5	24·8	19·4	5·4	·094	61	11·3		
29 30	.....	17·39†	14,750	23·1	18·9	4·2	·090	62	10·2	27·1	24·5	19·3	5·2	·095	62	11·7		
30 0	.....	17·35	14,790	21·9	18·9	3·0	·106	76	14·6	25·9	24·5	19·5	5·0	·097	64	12·2		
30 0	.....	17·33†	14,820															
31 0	39·8	17·31	14,870	23·1	19·3	3·8	·103	72	13·9	27·2								
32 0	39·0	17·26	14,930	21·9	18·5	3·4	·102	74	13·6	26·0								
32 30	.....	.....	.....	22·4	19·1	3·3	·106	75	14·6	26·5	24·0							
33 0	.....	17·16	15,090	21·6	18·9	2·7	·108	79	15·1	25·8	.....	.....	.....	.....	.....	.....	+ 12·0	·096
34 0	.....	17·12	15,130	20·6	18·2	2·4	·107	81	14·9	24·8	22·5							
35 0	.....	17·12	15,130															
35 20	.....	.....	.....	20·8	18·3	2·5	·107	80	14·9	25·1	22·2	.....	.....	.....	.....	.....	+ 6·0	·077
36 0	.....	17·17	15,060															
36 30	.....	.....	.....	22·2	18·9	3·3	·105	75	14·4	26·5								
37 0	.....	17·13	15,110	20·8	17·9	2·9	·103	78	13·9	25·1								
38 0																		
39 0	.....	16·92	15,410	.....	.....	.....	.....	.....	.....	.....	19·4	16·4	3·0	·090	71	10·2		
39 30	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	18·8							
40 0	.....	16·61	15,900	19·0	.....	.....	.....	.....	.....	.....	19·3	11·9	7·4	·054	43	− 3·2		
41 0	.....	16·41	16,180	16·3	10·5	5·8	·058	51	− 1·4	20·8	17·5	9·7	7·8	·044	38	− 8·4		
42 0	.....	16·26	16,400	14·6	8·7	5·9	·051	48	− 4·7	19·1	.....	.....	.....	.....	.....	.....	−10−	
43 0	.....	16·14	16,600	14·8	9·6	5·2	·059	55	− 0·9	19·4	19·5	10·8	8·7	·044	35	− 8·4	−15−	
43 40	.....	.....	.....	17·0	11·3	5·7	·062	54	+ 0·4	21·6	19·5	12·1	7·4	·056	44	− 2·3	−17·0	·032
44 0	.....	16·06	16,760															
45 0	.....	15·96	16,900	15·3	10·3	5·0	·063	58	+ 0·8	20·0								
46 0	.....	15·95	16,940	16·3	10·7	5·6	·061	54	− 0·1	21·0	18·3	11·7	6·6	·059	49	− 0·9		
46 10	.....	.....	.....	16·9	11·2	5·7	·062	54	+ 0·4	21·6								
46 20	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	15·6	9·4	6·2	·053	48	− 3·7		
47 0	.....	15·96	16,930															
48 0	.....	15·96	16,920	16·5	11·0	5·5	·062	55	+ 0·4	21·3	16·6	10·5	6·1	·057	50	− 1·8		
49 0	.....	.....	.....	13·4	8·6	4·8	·059	58	− 0·9	18·2	15·6	9·0	6·6	·050	46	− 5·2		
50 0	33·4	15·86	17,010	12·4	7·6	4·8	·056	57	− 2·3	17·3	15·1	8·5	6·6	·049	45	− 5·8		
51 0	.....	15·64	17,400	13·2	8·2	5·0	·056	56	− 2·3	18·1								
52 0	.....	15·46	17,660	10·6	6·7	3·9	·058	63	− 1·4	15·6	14·0	7·8	6·2	·049	47	− 5·8		
52 30	.....	.....	.....	10·2	6·8	3·4	·061	68	− 0·1	15·1								
53 0	.....	15·31	17,890															
54 0	.....	15·21	18,060	9·1	6·7	2·4	·066	77	+ 2·0	14·1								
55 0	.....	15·01	18,370	7·3	5·7	1·6	·068	85	2·8	12·4								

Table I. (Continued.)

Greenwich Mean Time.			Barometer. Therm.	Barometer. Reading corrected.	Height above sea-level.	Dry and Wet Thermometers, *aspirated*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Temperature corrected for change.	Dry and Wet Thermometers, *free*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Regnault's Hygrometer. Dew-point.	Tension of vapour.
	h	m	s	°	in.	feet	°	°	°	in.			°	°	°	in.		°	°	in.
August 26, 1852.	6	56	0	.....	14·91	18,590	9·5	6·5	3·0	·063	72	0·8	14·6							
		56	15	.....	14·88†	18,650														
		57	0	.....	14·76	18,870	9·9	6·7	3·2	·063	71	0·8	15·1	12·4	7·7	4·7	·058	59	— 1·4	
		57	30	.....	.....	.....	9·7	6·6	3·1	·063	72	0·8	14·9	12·4	7·4	5·0	·055	56	— 2·8	
		58	0	.....	14·65	19,070	10·1	7·1	3·0	·065	73	1·6	15·3	12·1	8·0	4·1	·062	64	+ 0·4	
		58	30	.....	14·68†	19,000														
		59	0	.....	14·61	19,100	8·3	6·3	2·0	·067	80	2·4	13·6							
		59	0	.....	14·66†	19,010														
	7	0	0	.....	14·69†	19,000	10·5	7·5	3·0	·066	73	2·0	15·8							
October 21, 1852.	2	47	0	.....	29·73	280	57·3	50·6	6·7	·305	64	44·1	57·3							
		48	0	.....	29·34	640	55·7	50·0	5·7	·309	68	44·5	55·7							
		48	30	62·2	29·12	850	54·8	49·6	5·2	·310	70	44·6	54·8							
		49	0	.....	28·97	990	54·3	49·1	5·2	·304	70	44·0	54·3	54·6	49·4	5·2	·308	71	44·4	
		49	30	.....	28·82	1,130	53·6	48·6	5·0	·301	71	43·7	53·6	53·7	48·7	5·0	·302	71	43·8	
		50	0	.....	28·50	1,440	52·4	48·0	4·4	·301	74	43·7	52·4	52·6	48·2	4·4	·303	74	43·9	
		51	0	.....	28·05	1,880	50·3	46·4	3·9	·288	76	42·5	50·3	50·3	46·5	3·8	·290	77	42·7	
		51	30	.....	.....	.....	49·8	46·0	3·8	·285	77	42·2	49·8							
		52	0	.....	27·61	2,310	49·3	46·1	3·2	·293	81	43·0	49·3	49·6	46·3	3·3	·294	80	43·1	
		53	0	.....	27·25	2,670	49·5	48·6	0·9	·346	94	47·8	49·5	50·0	48·7	1·3	·343	92	47·5	
		53	35	.....	.....	.....								50·4	49·4	1·0	·355	94	48·5	
		54	0	.....	27·10	2,820	51·0	50·0	1·0	·363	94	49·2	51·0	51·4	50·7	0·7	·375	96	50·1	
		55	0	.....	26·95	2,980	51·1	50·6	0·5	·376	97	50·2	51·1	51·7	51·4	0·3	·388	98	51·1	
		56	0	59·0	26·83	3,100	50·7	50·5	0·2	·378	99	50·4	50·7							
		56	30	.....	.....	.....								51·7	51·0	0·7	·379	96	50·5	
		57	0	.....	26·67	3,260	.....	.....	.....	.....	.....	.....	.....	51·6	50·9	0·7	·378	96	50·4	
		58	0	.....	26·52	3,420	50·1	50·1	0·0	·374	100	50·1	50·1	.....	.....	.....	.....	.....	.....	48·0	·349
	3	0	0	.....	26·19	3,760	50·7	50·1	0·6	·368	96	49·6	50·7							
		0	35	.....	.....	.....	51·3	50·2	1·1	·365	93	49·3	51·3							
		1	0	.....	25·96	4,010	51·7	50·1	1·6	·359	91	48·9	51·7	54·1	51·0	3·1	·355	83	48·5	
		1	30	60·6	25·84	4,140	52·3	49·9	2·4	·348	86	48·0	52·3							
		2	0	.....	25·73	4,260	.....	.....	.....	.....	.....	.....	.....	53·8	50·3	3·5	·343	81	47·5	
		2	30	.....	25·55	4,450	52·1	49·3	2·8	·337	84	47·0	52·2							
		3	0	.....	25·48	4,520	52·0	49·1	2·9	·334	83	46·8	52·1	52·6	49·1	3·5	·328	80	46·2	
		4	0	.....	25·07	4,960	50·3	45·9	4·4	·282	75	41·9	50·4	52·0	46·4	5·6	·276	69	41·3	
		4	35	.....	.....	.....	49·9	45·0	4·9	·268	72	40·4	50·0	51·4	45·6	5·8	·266	68	40·2	
		5	0	58·5	24·70	5,370	49·6	44·5	5·1	·261	71	39·7	49·8	50·6	45·0	5·6	·261	68	39·7	
		5	30	.....	.....	.....	48·9	44·0	4·9	·258	72	39·3	49·1							
		6	0	.....	24·38	5,710	48·1	43·5	4·6	·255	73	39·0	48·3	49·2	43·7	5·5	·249	69	38·3	
		6	30	.....	.....	.....	47·3	42·8	4·5	·250	73	38·4	47·5							
		6	40	.....	24·08	6,050	.....	.....	.....	.....	.....	.....	.....							
		7	0	.....	23·95	6,190	46·6	42·3	4·3	·247	74	38·1	46·8							
		7	30	.....	.....	.....	46·3	42·0	4·3	·245	75	37·9	46·5							
		8	0	.....	23·46	6,750	44·9	40·6	4·3	·231	74	36·2	45·2	45·9	41·1	4·8	·232	72	36·3	
		9	0	.....	23·29	6,940	44·5	39·0	5·5	·206	67	32·9	44·8							
		10	0	55·0	22·92	7,380	44·3	37·9	6·4	·189	61	30·5	44·6	.....	.....	.....	.....	.....	.....	27·0	·167
		11	0	.....	22·73	7,590	42·9	36·2	6·7	·173	59	28·1	43·3							
		11	30	53·6	22·49	7,880	42·4	34·7	7·7	·153	53	24·7	42·8							
		12	0	.....	22·31	8,090	41·6	34·2	7·4	·152	55	24·5	42·0							
		12	40	.....	.....	.....	40·8	33·8	7·0	·153	57	24·7	41·2	42·7	34·8	7·9	·153	53	24·7	
		13	0	.....	22·07	8,370	40·0	32·9	7·1	·146	55	23·4	40·4	39·6	32·7	6·9	·146	56	23·4	
		13	30	.....	.....	.....	39·6	32·4	7·2	·142	55	22·6	40·1							
		14	30	.....	21·46	9,110	37·8	30·8	7·0	·139	57	22·0	38·3							
		15	0	.....	21·31	9,290	36·7	30·6	6·1	·145	62	23·2	37·2	36·7	30·4	6·3	·142	60	22·6	
		16	0	.....	21·08	9,550	35·7	30·9	4·8	·157	69	25·4	36·2	37·1	31·2	5·9	·151	63	24·3	
		17	0	.....	20·78	9,950	34·5	29·8	4·7	·150	69	24·1	35·1	35·6	30·1	5·5	·147	65	23·6	
		17	30	.....	.....	.....	33·8	29·4	4·4	·150	71	24·1	34·4							
		18	0	.....	20·54	10,240	32·7	28·9	3·8	·151	74	24·3	33·3	33·9	28·7	5·2	·140	66	22·2		
		18	30	.....	.....	.....	32·1	28·4	3·7	·149	74	23·9	32·7	.....	.....	.....	.....	.....	.....	20·0	·129

Table I. (Continued.)

Greenwich Mean Time.			Barometer. Therm.	Barometer. Reading corrected.	Height above sea-level.	Dry and Wet Thermometers, *aspirated*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Temperature corrected for change.	Dry and Wet Thermometers, *free*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Regnault's Hygrometer. Dew-point.	Tension of vapour.
	h	m	°	in.	feet	°	°	°	in.		°	°	°	°	in.			°	in.	
October 21, 1852.	3	19	0	20·22	10,650	31·7	27·6	4·1	·142	72	22·6	32·3								
		19	30			31·2	27·0	4·2	·138	71	21·8	31·8								
		20	0	20·03	10,900	31·1	26·2	4·9	·129	67	20·0	31·8	31·0	26·3	4·7	·130	68	20·2		
		21	0	19·85	11,130	29·8	25·7	4·1	·131	71	20·4	30·5	31·6	26·8	4·8	·132	67	20·6		
		22	0	19·63	11,420	29·4	25·2	4·2	·128	71	19·8	30·1	30·4	25·9	4·5	·130	69	20·2		
		23	0	19·41	11,700	27·8	24·1	3·7	·126	73	19·3	28·6	29·2	24·7	4·5	·123	68	18·7		
		24	0	19·23	11,940	26·8	23·2	3·6	·121	73	18·2	27·6								
		25	0	19·10	12,130	26·8	22·6	4·2	·115	69	16·8	27·6	28·2	23·8	4·4	·119	70	17·8		
		26	0	18·95	12,320	25·9						26·8							17·0	·116
		27	0	18·85	12,450	24·9						25·8	27·5	24·2	3·3	·129	76	20·0		
		28	0	18·73	12,630	25·9						26·8								
		28	30			25·8	22·0	3·8	·114	71	16·6	26·7								
		29	0	18·72	12,640	25·7	22·0	3·7	·115	72	16·8	26·7	27·6	24·3	3·3	·130	76	20·2		
		29	30			26·0	22·0	4·0	·113	70	16·4	27·0								
		30	0			26·9	23·0	3·9	·119	72	17·8	27·9	27·7	24·1	3·6	·127	74	19·5	19·0	·124
		30	25	18·85	12,470															
		31	25	19·15	12,050	26·5	23·4	3·1	·126	77	19·3	27·5	27·8	24·2	3·6	·127	74	19·5	18·0	·120
		33	50	19·97	10,960	29·7	25·7	4·0	·131	71	20·4	30·8							19·0	·124
		36	0	20·67	10,070	33·4	29·2	4·2	·150	72	24·1	34·6								
		37	30	21·24	9,360	35·5	30·2	5·3	·149	66	23·9	36·7	36·2	31·3	4·9	·159	69	25·7		
		38	0	21·50	9,040	36·5	30·2	6·3	·140	60	22·2	37·7								
		39	0	21·81	8,660								38·6	32·3	6·3	·149	59	23·9		
		39	30	21·92	8,530	38·4	31·0	7·4	·136	55	21·4	39·7								
		40	0	22·05	8,380	39·3	31·2	8·1	·132	51	20·6	40·6								
		41	0	22·25	8,140	40·8	31·5	9·3	·125	46	19·1	42·1							25·0	·155
		42	30	22·46	7,900	42·5						43·8							24·0	·149
		43	30	22·55	7,800	43·2	35·1	8·1	·153	52	24·7	44·6	43·4	37·7	5·7	·195	66	31·4		
		45	0	22·49	7,870	42·5	35·2	7·3	·160	55	25·9	43·9								
		46	0	22·40	7,970	42·4	34·7	7·7	·153	53	24·7	43·8								
		48	0	22·31	8,090	42·9	34·7	8·2	·149	51	23·9	44·4							20·0	·129
		49	0	22·34	8,060	43·7	36·3	7·4	·169	56	27·4	45·2	43·1	35·7	7·4	·164	56	26·6		
		50	0	22·55	7,810	44·3	35·9	8·4	·156	51	25·2	45·7	44·6	37·3	7·3	·177	57	28·7		
		53	0	23·42	6,780	46·1	39·8	6·3	·206	63	32·9	47·7	46·6	40·6	6·0	·216	65	34·3		
		54	0	23·62	6,550	46·4	41·2	5·2	·229	69	35·9	48·0								
		54	20										46·5	41·5	5·0	·233	70	36·4		
		57	30	47·5	23·70	6,450														
		58	30		23·74	6,400	44·7	40·8	3·9	·236	76	36·8	46·5						35·0	·222
	4	1	30		24·62	5,420	46·7	42·4	4·3	·247	74	38·1	48·5							
		3	30		25·51	4,460	49·4	45·6	3·8	·284	78	42·1	51·3							
		4	0		25·75	4,210								49·5	46·2	3·3	·295	80	43·2	
		5	0		26·12	3,820	50·6	47·6	3·0	·314	82	45·0	52·5	50·6	48·1	2·5	·325	85	46·0	
		5	45		26·47	3,450	50·6	48·6	2·0	·335	88	46·8	52·5	50·6	48·7	1·9	·338	89	47·1	
		6	30	51·0	26·73	3,180	50·2	48·7	1·5	·342	91	47·5	52·1							
		8	0		27·03	2,880	50·1	49·1	1·0	·352	94	48·3	52·1							
November 10, 1852.	2	22	30	52·0	29·85	240	49·6	44·3	5·3	·247	67	38·1	49·6							
		23	0		29·55	510														
		23	30		29·47	580	48·2	43·5	4·7	·245	70	37·9	48·2	48·1	43·2	4·9	·240	69	37·3	
		24	30		28·96	1,060	46·4	42·2	4·2	·239	72	37·1	46·4	46·3	41·9	4·4	·233	71	36·4	
		25	0	52·0	28·66	1,330	45·0	41·6	3·4	·242	77	37·5	45·0							
		25	30		28·35	1,630	43·6	40·8	2·8	·241	80	37·4	43·6							
		26	10		27·99	1,970	42·3	38·7	3·6	·214	75	34·0	42·3	42·0	38·0	4·0	·203	72	32·5	
		27	10		27·18	2,760	39·6	37·4	2·2	·218	83	34·5	39·6	39·5	37·1	2·4	·214	82	34·0	
		28	0		26·78	3,150	38·0	36·5	1·5	·218	88	34·5	38·0	37·8	36·3	1·5	·217	89	34·4	
		29	0		26·23	3,700	37·2	36·2	1·0	·221	92	34·9	37·2	37·2	36·2	1·0	·221	92	34·9	
		29	30		25·93	4,010	36·8	36·0	0·8	·222	94	35·1	36·8							
		30	40		25·40	4,560	35·5	34·7	0·8	·211	94	33·6	35·5						33·0	·206
		31	0		25·08	4,890	34·3	34·0	0·3	·211	98	33·6	34·3							
		32	0		24·65	5,350	33·7	33·4	0·3	·206	97	32·9	33·7	33·8	33·4	0·4	·206	97	32·9	

Table I. (Continued.)

Greenwich Mean Time.	Barometer. Therm.	Barometer. Reading corrected.	Height above sea-level.	Dry and Wet Thermometers, *aspirated*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Temperature corrected for change.	Dry and Wet Thermometers, *free*. Dry.	Wet.	Diff.	Tension of vapour.	Relative humidity.	Dew-point.	Regnault's Hygrometer. Dew-point.	Tension of vapour.
November 10, 1852. h m s	°	in.	feet	°	°	°	in.		°		°	°	°	in.		°	°	in.
2 33 10	.....	24·17	5,880	34·7	33·4	1·3	·197	90	31·7	34·7								
34 0	44·3	23·94	6,140	35·6	31·4	4·2	·160	71	25·9	35·6								
34 30	.....	23·76	6,350	35·8	30·8	5·0	·150	66	24·1	35·8	36·3	31·8	4·5	·161	69	26·1		
35 0	.....	23·63	6,500	35·8	30·8	5·0	·150	66	24·1	35·8								
35 30	.....	23·45	6,700	35·6	31·4	4·2	·161	71	26·1	35·6								
36 30	43·0	23·12	7,070	34·7	31·2	3·5	·167	76	27·1	34·7	35·1	31·4	3·7	·166	75	26·9		
37 30	.....	22·72	7,530	33·9	31·6	2·3	·178	84	28·9	33·9	33·7	31·3	2·4	·176	83	28·5		
38 0	.....	.....	.....	32·7	31·0	1·7	·179	88	29·0	32·7								
38 30	.....	22·29	8,030	32·7	31·4	1·3	·186	91	30·1	32·7								
39 0	.....	21·97	8,420	32·5	31·4	1·1	·187	92	30·2	32·5	32·2	31·2	1·0	·186	93	30·1		
40 0	.....	21·58	8,890	31·4	29·7	1·7	·171	88	27·7	31·4	31·3	29·7	1·6	·172	89	27·9		
41 0	40·5	21·07	9,510	29·8	27·2	2·6	·149	81	23·9	29·8								
42 0	.....	20·71	9,970	29·5	26·2	3·3	·138	76	21·8	29·5	28·8	25·9	2·9	·139	78	22·0		
43 0	.....	20·17	10,630	26·9	24·2	2·7	·132	79	20·6	26·9	27·2	24·1	3·1	·129	77	20·0		
44 0	.....	19·65	11,310	24·9	22·4	2·5	·124	80	18·9	24·9	.....	.....	.....	.....	.....	.....	16·0	·111
45 30	.....	18·74	12,500	20·3	(32)	.....	.....	.....	.....	20·3	20·6	(32)	.....	.....	.....	.....	10·0	·089
46 30	36·0	18·30	13,080	17·4	.....	.....	.....	.....	.....	17·4								
47 0	.....	17·90	13,650	16·3	.....	.....	.....	.....	.....	16·3	16·8							
47 30	.....	17·75	13,860	15·9	12·4	3·5	·076	68	5·8	15·9								
49 0	.....	16·66	15,480	13·4	8·9	4·5	·060	59	− 0·5	13·4								
49 40	29·5	16·44	15,820	12·8	8·1	4·7	·057	58	− 1·8	12·8								
50 30	.....	16·23	16,150	.....	.....	.....	.....	.....	.....	.....	12·7	8·4	4·3	·060	61	− 0·5		
51 0	.....	15·96	16,580	11·9	10·3	1·6	·081	84	+ 7·4	11·9								
52 0	.....	15·28	17,620	7·0	5·6	1·4	·068	85	+ 2·8	7·0	9·1	5·9	3·2	·060	69	− 0·5		
52 30																		
54 0	.....	14·72	18,520	3·8	1·3	2·5	·052	74	− 4·2	3·8								
55 0	.....	14·44	18,990	2·7	0·6	2·1	·052	77	− 4·2	2·7	4·2	0·4	3·8	·043	60	− 9·1	− 30(?)	·019
56 0	.....	14·24	19,330	1·5	− 0·1	1·6	·053	82	− 3·7	1·5								
57 0	.....	14·01	19,730	0·4	− 0·5	0·9	·056	90	− 2·3	0·4	1·1	− 0·8	1·9	·050	78	− 5·2		
58 0	.....	14·04	19,690	1·0	+ 0·2	0·8	·058	91	− 1·4	1·0								
59 0	.....	14·20	19,400	1·5	0·0	1·5	·054	83	− 3·2	1·5	2·6	+ 1·1	1·5	·056	83	− 2·2		
3 1 0	.....	14·78	18,430	4·4	+ 0·9	3·5	·046	64	− 7·4	4·4	5·2	2·5	2·7	·053	71	− 3·7		
2 0	18·2	14·95	18,110	3·4	− 0·2	3·6	·042	60	− 9·7	3·4								
3 30	.....	14·94	18,130	3·4	− 1·8	5·2	·031	44	−18·0	3·4	4·8	− 0·2	5·0	·035	48	−14·0		
5 0	.....	14·60	18,700	2·2	− 2·1	4·3	·035	53	−14·0	2·2	3·9	− 0·9	4·8	·035	49	−14·0		
6 0	.....	14·40	19,020	0·5	.....	.....	.....	.....	.....	0·5	3·0	− 1·0	4·0	·039	57	−12·0		
7 0	14·0	14·08	19,560	− 1·3	.....	.....	.....	.....	.....	− 1·3								
8 30	11·5	13·60	20,430	− 2·3	.....	.....	.....	.....	.....	− 2·3	.....	.....	.....	.....	.....	.....	−22·0	·027
9 0	.....	13·45	20,650	− 5·1	.....	.....	.....	.....	.....	− 5·1								
10 30	.....	13·02	21,380	− 9·3	.....	.....	.....	.....	.....	− 9·3								
12 0	.....	12·65	22,110	− 9·6	.....	.....	.....	.....	.....	− 9·6								
13 0	.....	12·51	22,370	−10·5	.....	.....	.....	.....	.....	−10·5								
14 30	.....	12·40	22,640	− 8·9	.....	.....	.....	.....	.....	− 8·9								
15 0	.....	12·24	22,930															
16 0	.....	12·24	22,930															
17 0	.....	.....	.....	− 6·5	.....	.....	.....	.....	.....	− 6·5								
19 0	.....	12·94	21,640	− 5·3	.....	.....	.....	.....	.....	− 5·3								

§ 5. *Variation of Temperature with Height.*

The observations of temperature given in the preceding table, with the corresponding heights, have been divided into groups, each group being composed of the observations within 1000 feet*. The numbers employed are those in column 11 of the table, which have been corrected for the change occurring in the temperature during the continuance of the experiments, as given by corresponding observations at the earth's surface. This correction is very probably inaccurate to some extent; but our information is as yet so imperfect with regard to the diurnal variations of temperature in the upper parts of the atmosphere, that no other course has appeared open to me. Any error arising from this cause is probably small in any of the series now under consideration, with the exception perhaps of August 26, when the hourly changes, as well as the time occupied in the ascent, were considerable. These groups are contained in the following table:—

TABLE II.—Means of Groups of the Observations of Temperature at different heights in the four Balloon Ascents of 1852, with the differences between the observed temperatures and those calculated by equations (1.) and (2.) from each whole series, and from the adopted divisions of each series.

Date.	Groups.			Temperature, observed − calculated.					
				Whole Series.		Lower Division.		Upper Division.	
	No. of obs.	Height.	Temperature.	By eq. (1.).	By eq. (2.).	By eq. (1.).	By eq. (2.).	By eq. (1.).	By eq. (2.).
		feet		°	°	°	°	°	°
August 17	(1)	(120)	(71·2)	−1·4	+1·3	0·0	.....	(−5·1)	(−7·5)
	1	2,440	62·8	−2·4	−1·4	0·0	.....	(−5·6)	(−7·3)
	1	3,460	59·2	−2·8	−2·4	0·0	.....	(−5·8)	(−7·1)
	1	4,110	58·1	−1·9	−1·7	(+1·3)	.....	(−4·7)	(−5·8)
	1	5,880	57·8	+3·2	+2·8	(+7·3)	.....	(+0·9)	(+0·3)
	3	6,800	54·0	+3·2	+1·6	(+6·8)	.....	(+0·2)	(−0·2)
	5	7,530	51·4	+2·0	+1·1	(+6·9)	.....	+0·1	−0·2
	8	8,550	49·0	+2·7	+1·7	(+8·1)	.....	+1·1	+1·0
	7	9,470	44·4	+1·0	−0·1	(+6·9)	.....	−0·4	−0·4
	4	10,680	40·4	+0·7	−0·4	(+7·2)	.....	−0·3	−0·2
	4	11,620	37·2	+0·4	−0·6	.....	.....	−0·3	−0·1
	8	12,250	34·9	+0·1	−0·9	.....	.....	−0·5	−0·3
	3	13,480	30·8	−0·2	−0·9	.....	.....	−0·4	−0·2
	4	14,550	27·0	−0·7	−1·1	.....	.....	−0·6	−0·4
	6	15,510	24·4	−0·3	−0·4	.....	.....	0·0	+0·1
	10	16,600	20·6	−0·8	−0·4	.....	.....	−0·1	−0·1
	6	17,440	19·6	+0·9	+1·6	.....	.....	+1·7	+1·7
	8	18,490	15·0	−0·5	+0·9	.....	.....	+0·6	+0·5
	8	19,320	10·5	−2·4	−0·5	.....	.....	−1·0	−1·3

* The third group of October 21 extends only from 2000 to 2670 feet; the two observations between the latter height and 3000 feet, showing a marked change which refers them more intimately to the succeeding group. The lowest group in each series depends solely upon observations taken in the car, with the exception of that of August 17, when no observations having been recorded below 2000 feet, the general temperature at the earth has been adopted as the first result.

TABLE II. (Continued.)

Date.	Groups.			Temperature, observed − calculated.					
	No. of obs.	Height.	Temperature.	Whole Series.		Lower Division.		Upper Division.	
				By eq. (1.).	By eq. (2.).	By eq. (1.).	By eq. (2.).	By eq. (1.).	By eq. (2.).
		feet.	°	°	°	°	°	°	°
August 26	3	700	64·8	+1·6	+2·3	− 0·2	+ 0·2		
	2	1,380	62·2	+0·8	+1·4	− 0·4	− 0·3		
	5	2,480	59·1	+0·6	+0·9	+ 0·4	+ 0·2		
	4	3,390	55·7	−0·4	−0·3	+ 0·3	0·0		
	4	4,430	51·7	−1·7	−1·7	0·0	− 0·3	(− 9·8)	(−12·3)
	3	5,620	48·0	−2·3	−2·5	+ 0·5	+ 0·5	(− 9·5)	(−11·4)
	3	6,350	44·3	−4·1	−4·3	− 0·6	− 0·3	(−10·8)	(−12·3)
	3	7,390	41·9	−3·7	−4·1	(+ 0·7)	(+ 1·6)	(− 9·7)	(−10·8)
	3	8,730	42·9	+0·8	+0·4	(+ 6·4)	(+ 8·3)	(− 4·2)	(− 4·9)
	10	9,510	42·3	+2·2	+1·8	(+ 8·6)	(+11·2)	(− 2·2)	(− 2·1)
	20	10,590	41·0	+3·7	+3·3	(+11·1)	(+14·9)	+ 0·2	− 0·1
	25	11,630	37·4	+2·9	+2·5	(+11·2)	(+16·3)	0·0	0·0
	46	12,490	34·0	+1·7	+1·4	(+10·9)	(+17·1)	− 0·5	− 0·4
	5	13,350	31·9	+1·9	+1·6	.....	.....	+ 0·3	+ 0·4
	11	14,500	28·1	+1·1	+1·0	.....	.....	+ 0·4	+ 0·6
	7	15,200	25·3	+0·1	+0·1	.....	.....	− 0·1	+ 0·1
	8	16,700	20·2	−1·1	−0·8	.....	.....	− 0·1	− 0·1
	4	17,460	16·5	−2·8	−2·3	.....	.....	− 1·3	− 1·3
	8	18,750	14·5	−1·4	−0·6	.....	.....	+ 1·0	+ 0·8
October 21.....	4	690	55·5	−1·4	+2·1	+ 0·2	.....	(−10·9)	(−10·1)
	3	1,480	52·1	−3·1	−1·0	− 0·4	.....	(−11·7)	(−11·0)
	3	2,360	49·5	−3·6	−3·1	+ 0·2	.....	(−11·3)	(−10·8)
	10	3,330	51·4	+0·5	−0·3	(+ 5·5)	.....	(− 6·1)	(− 5·8)
	6	4,420	51·7	+3·3	+1·5	(+ 9·8)	.....	(− 2·1)	(− 1·9)
	6	5,530	48·9	+3·1	+0·6	(+10·9)	.....	− 1·2	− 1·1
	7	6,580	46·5	+3·1	+0·5	(+12·3)	.....	− 0·1	0·0
	8	7,770	44·1	+3·4	+1·1	(+14·1)	.....	+ 1·6	+ 1·5
	9	8,280	41·7	+2·2	+0·2	(+13·6)	.....	+ 0·9	+ 0·8
	6	9,380	36·9	−0·1	−1·1	(+12·7)	.....	− 0·2	− 0·3
	8	10,520	32·7	−1·7	−1·2	.....	.....	− 0·6	− 0·6
	4	11,550	29·2	−2·8	−0·5	.....	.....	− 0·6	− 0·6
	9	12,430	27·0	−3·0	+1·1	.....	.....	+ 0·2	+ 0·2
November 10...	2	410	48·9	−0·1	+4·7	+ 0·2	.....	(− 9·3)	(+ 1·5)
	4	1,500	44·3	−2·0	+1·4	− 0·3	.....	(−10·6)	(− 1·4)
	1	2,760	39·6	−3·5	−1·6	− 0·2	.....	(−11·4)	(− 3·9)
	2	3,430	37·6	−3·9	−2·6	+ 0·3	.....	(−11·4)	(− 4·7)
	3	4,490	35·5	−3·3	−3·0	(+ 1·2)	.....	(−10·3)	(− 4·8)
	2	5,620	34·2	−1·8	−2·4	(+ 5·1)	.....	(− 8·1)	(− 3·8)
	4	6,420	35·7	+1·7	+0·5	(+ 9·6)	.....	(− 4·2)	(− 0·6)
	3	7,460	33·8	+2·4	+0·7	(+11·6)	.....	(− 2·9)	(− 0·3)
	3	8,440	32·2	+3·2	+1·1	(+13·7)	.....	(− 2·5)	(+ 0·4)
	2	9,740	29·6	+3·9	+1·3	.....	.....	− 0·2	+ 0·9
	1	10,630	26·9	+3·4	+0·7	.....	.....	− 0·2	+ 0·4
	1	11,300	24·9	+3·1	+0·3	.....	.....	− 0·1	+ 0·1
	1	12,500	20·3	+1·5	−1·2	.....	.....	− 1·1	− 1·3
	3	13,530	16·5	+0·3	−2·3	.....	.....	− 1·7	− 2·3
	2	15,650	13·1	+2·1	+0·4	.....	.....	+ 1·3	+ 0·5
	1	16,580	11·9	+3·3	+2·0	.....	.....	+ 3·0	+ 2·2
	1	17,620	7·0	+1·0	+0·4	.....	.....	+ 1·2	+ 0·5
	6	18,480	3·3	−0·6	−0·4	.....	.....	+ 0·1	− 0·4
	6	19,460	0·6	−0·8	+0·2	.....	.....	+ 0·4	+ 0·2
	2	20,540	− 3·7	−2·4	−0·3	.....	.....	− 0·6	− 0·4
	2	21,510	− 7·3	−3·6	−0·4	.....	.....	− 1·2	− 0·6
	3	22,370	− 9·7	−3·9	+0·5	.....	.....	− 1·0	+ 0·1

In order to deduce from these numbers an approximation to the normal progression of temperature, freed from accidental irregularities, each series was in the first instance arranged in equations of the form—

$$T=X+YH, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

T being the observed temperature at the height H; Y the change in degrees of temperature due to 1000 feet of height; and X the temperature at the level of the sea, which, with the addition of the quantity YH, would best represent the observed temperatures throughout the series, on the supposition of the change being uniform with the height. X and Y were eliminated by the method of least squares, and the following values obtained for the different series:—

	Aug. 17.	Aug. 26.	Oct. 21.	[Nov. 10.
X=	72°·76	64°·98	58°·49	50°·02
Y=	− 3·097	− 2·617	− 2·291	− 2·496
Mean error .	1·72	2·16	2·65	2·56

On the supposition that the rate of change is not constant, but that it varies with the height, the following interpolating equation was employed,—

$$T=x+yH+zH^2+ \&c., \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

omitting higher powers of H than the second. When the same method of elimination was adopted, the following values were found:—

	Aug. 17.	Aug. 26.	Oct. 21.	Nov. 10.
x=	70°·17	64°·11	53°·36	44°·69
y=	− 2·363	− 2·346	+ 0·1132	− 1·095
z=	− 0·03613	− 0·01424	− 1·868	− 0·06070
Mean error	1·30	2·13	1·35	1·71

In Table II. will be found the differences between the observed temperatures, and those resulting from the two forms of equation employed. The progress of those differences in each series seems to follow a distinct law; there being in all cases a maximum of negative differences at a short distance from the earth, varying from about 2500 feet on October 21, to 6000 feet on August 26, followed also in each by a maximum of positive differences at an additional height of 3000 to 5000 feet. This peculiar departure from a regular progression will be distinctly traced in the projected results (Plates XIX.—XXII.). It is there seen, in all the four series, that after a steady decrease of the temperature in the lower portion of the curve, this decrease becomes arrested, and, for a space of about 2000 feet, the temperature remains almost constant, or even increases by a small amount; the decrease being afterwards resumed and continued, without much variation, throughout the upper portion of the curve. In the series of August 17 and 26 this fact is strikingly coincident with a large and abrupt diminution in the amount of aqueous vapour; the same coincidence being

exhibited, in a less marked manner, on November 10. On October 21, the departure from a uniform decrease is very decidedly shown in connection with the stratum of dense cloud passed through. The temperature had been uniformly decreasing until the thick cloud was reached, when a decided rise commenced, which continued through the cloud, and for a space of about 600 feet above it; after which height the decrease was resumed, at first slowly, and afterwards with more rapidity.

The disturbance in the variation of the temperature now noticed, is in each series exhibited in such a systematic manner, that the hypothesis of a regular progression at all heights can scarcely be maintained. In order therefore to arrive at some approximate value of the normal variation of temperature in the atmosphere, it appears necessary to make abstraction of the disturbing cause. This I have endeavoured to do by dividing each series into two divisions; 1st, between the earth and the height where the diminution of temperature appears to be arrested; 2nd, above the point where the regular diminution of temperature seems to be resumed, omitting the space which is under the influence of the disturbance. The divisions adopted for the four series are as follows:—

	Aug. 17. Feet.	Aug. 26. Feet.	Oct. 21. Feet.	Nov. 10. Feet.
Lower division	0 to 4000	0 to 7000	0 to 2700	0 to 4000
Upper division	7000 to 20,000	10,000 to 19,000	5000 to 13,000	9000 to 23,000

These partial series have been examined by the same methods as the entire series; the number of groups in the lower division being, however, with the exception of August 26, too small to admit of the application with any advantage of equation (2.). The results for the different series are as follows:—

		By Equation (1.).			By Equation (2.).			
		X	Y	Mean error.	x	y	z	Mean error.
Aug. 17.	Lower division	71°·62	−3°·598	0°·00				
	Upper division	76·68	−3·371	0·73	79°·17	−3°·771	+0°·01484	0°·71
Aug. 26.	Lower division	67·46	−3·549	0·39	66·75	−2·969	−0·08220	0·30
	Upper division	76·36	−3·355	0·60	81·68	−4·104	+0·02552	0·58
Oct. 21.	Lower division	57·77	−3·581	0·26				
	Upper division	68·77	−3·376	0·82	67·81	−3·162	−0·01119	0·80
Nov. 10.	Lower division	50·21	−3·760	0·26				
	Upper division	59·45	−3·046	1·22	48·05	−1·516	−0·04791	1·02

The values of the constants in equation (2.), deduced from the higher divisions, show that, in the two series of August 17 and 26, the temperature decreases *less* rapidly as we ascend; whilst the values for October 21 and November 10 indicate a contrary result. The value of the second term (z) is, with the exception of the series of November 10, very small, and the amounts of the mean errors show that the observations are little better represented than by the single constant of equation (1.).

On the whole, we are scarcely at liberty to conclude from these results that the progress of the temperature, when free from disturbing influences, is other than *uniform* with the height.

Confining our attention now to the results deduced by equation (1.), we infer from them that in each series the rate of decrease of temperature, *below* the stratum where the disturbing influence exists, is *greater* than above that stratum; the ratio of the rate of decrease in the lower division to that in the higher, being—

On Aug. 17, 1·067; on Aug. 26, 1·058; on Oct. 21, 1·061; and on Nov. 10, 1·234.

If, in order to obtain the mean rate of decrease of temperature in the atmosphere, freed from disturbing causes, we allow to the lower and upper series values proportional to the spaces within which the observations for each division occur, we have the following numbers representing the decrease of temperature for 1000 feet of height:—

Aug. 17	3°·434	Oct. 21 . . .	3°·431
Aug. 26	3·440	Nov. 10 . . .	3·205

the values for the first three series being almost identical; that for the fourth differing from them by $\frac{1}{18}$th of the whole.

It may be convenient to give here the results for the rate of diminution of temperature obtained by different methods, expressed in the form usually adopted by meteorologists, viz. the height in feet equivalent to a decrease of one degree FAHRENHEIT.

	Aug. 17.	Aug. 26.	Oct. 21.	Nov. 10.
From the whole series	322·9	382·0	436·5	400·6
From first and last groups only . .	316·3	358·8	411·9	374·7
From lower division	277·9	281·8	279·3	266·0
From upper division	296·5	298·1	296·2	328·3
Mean of two divisions	292·0	290·7	291·5	312·0

The amount of distortion in the curve representing the diminution of temperature, produced by the disturbing influence which has been noticed, may be approximately stated at 7° on Aug. 17; 10°$\frac{1}{2}$ on Aug. 26; 11°$\frac{1}{2}$ on Oct. 21, and 12° on Nov. 10.

§ 6. *Variation of the Hygrometric Condition of the Air.*

As the amount of aqueous vapour in the air must necessarily decrease with the temperature, even although the proportion to the whole capacity of the air for moisture should remain constant, the changes at different heights may probably be most conveniently studied by examining the results for the "Relative Humidity," or the proportion which the amount of vapour present in the air bears to that which it would contain were it completely saturated. Since these changes do not, as in the case of temperature, appear to follow any regular course from which normal results might be derived, I shall here only state briefly the most prominent peculiarities pre-

sented by each series, referring for further information to the table of observations and to the projected results.

August 17.—We see by the curve of relative humidity for this day, that, from the earth's surface to the height of about 4000 feet, the humidity slightly increased; the presence of a considerable quantity of moisture being also shown by the existence of a partial stratum of cloud at the height of about 2500 feet. Between the heights of 4000 and 5880 the humidity decreased *with great rapidity* from about 85 to less than 35. For a considerable space little alteration took place, with the exception of a sudden increase at the height of about 9000 feet, which was confined to a stratum of not more than 400 feet; but as the evidence of its existence depends upon only one or two observations it may perhaps be doubtful. From 10,000 feet to 12,300, the humidity gradually increased to about 90, which value it retained very constantly through fully 4000 feet. After 16,500 feet there were considerable irregularities, there being however a comparatively dry stratum between 18,000 and 19,000 feet, which was followed by a decided increase in the humidity. These indications agree well with what is stated in § 3. with regard to the occasional existence of cloud above the height of 13,000 feet, and with the fact that at the highest point reached a mass of cloud was seen at a short distance above. In this series we can trace the existence of two distinct strata of moist air, besides a third, which undoubtedly existed at a greater height, but which was not quite reached.

August 26.—As on the first ascent, the humidity steadily increased from the earth's surface. Between the heights of 7200 and 8950 feet it also *rapidly* diminished from 92 to 26. For some distance the variations were no greater than might be supposed to arise from uncertainty of observation in such extreme circumstances. It will be remarked, on examining the curve of the tension of vapour, that whilst the indications of Regnault's hygrometer did not differ much from those of the wet thermometer at the height of about 11,000 feet, the difference became considerable at about 12,000 or 13,000 feet; thus rendering it probable that at the latter heights the relative humidity, as deduced from the dry and wet thermometers, was too great. The general accordance between the two hygrometers was however nearly restored at about 15,000 feet, confirming the rise which there took place in the amount of vapour. We may therefore consider that there was little change in the humidity from 9000 to 14,000 feet, a decided increase having however occurred at 15,000 feet, followed by a diminution till 16,400 feet; an increase having been again indicated in the remainder of the curve. The principal stratum of vapour on this day extended from the earth to 7200 feet, a second and perhaps a third of smaller thickness existing at 15,000 and 18,000 feet.

October 21.—The amount of moisture in the air on this occasion was considerable. The relative humidity increased as we left the earth, at first slowly till the height of 2000 feet, when irregular masses of cloud became frequent, and afterwards with more rapidity, till within the principal cloudy mass, at a height of 3450 feet, it attained the

point of complete saturation. After leaving the cloud the humidity diminished steadily but not very rapidly till 5300 feet, where a slight rise commenced, continuing till 6700 feet; it then decreased till 8300 feet, when it rose again and remained nearly constant at 70 for the last 3000 feet of the ascent. The changes occurring in this series were neither to the same extent nor so abrupt in their character as those shown in the first two.

November 10.—The humidity, again, as in all the previous series, increased from the earth to the first cloud, which was at a low elevation and of but little density; upon leaving it, at about 1900 feet, a slight depression took place. Immediately above this low cloud a different current of air existed, shortly after entering which the humidity again increased until, in the second cloud, it became nearly complete; the decrease, after leaving the cloud at 5000 feet, becoming rapid and attaining a minimum at 6500 feet. A second well-defined maximum was reached at 8300 feet, followed at 10,000 feet by a secondary minimum. The humidity diminished on the whole till about 15,800 feet, when a sudden increase commenced, which continued from 16,500 to 17,600 feet, followed by an equally sudden decrease at 18,000 feet, the humidity subsequently increasing. The fluctuations in this series were numerous, there having been no fewer than four or perhaps five different strata of vapour.

§ 7. *General Remarks.*

The principal results deduced from the experiments described may be thus generally stated.

The temperature of the air decreases uniformly with the height above the earth's surface, until at a certain elevation, varying on different days, the decrease is arrested, and for a space of from 2000 to 3000 feet the temperature remains nearly constant, or even increases by a small amount; the regular diminution being afterwards resumed and generally maintained, at a rate slightly less rapid than in the lower part of the atmosphere, and commencing from a higher temperature than would have existed but for the interruption noticed. This interruption in the decrease of temperature is accompanied by a large and abrupt fall in the temperature of the dew-point, or by actual condensation of vapour, from which it may be inferred that the disturbance in the progression of temperature arises from a development of heat in the neighbourhood of the plane of condensation. The subsequent falls in the temperature of the dew-point are generally of an abrupt character, and corresponding interruptions in the decreasing progression of temperature are sometimes distinguishable, but in a less degree; as might indeed be expected from the fact, that at lower temperatures the variations in the absolute amount of aqueous vapour are necessarily smaller, and their thermic effects consequently diminished.

Dr. MILLER'S *Analysis of Air collected in the Ascents.*

"King's College, London, 5 May, 1853.

"MY DEAR SIR,—The following particulars of my examinations of some of the specimens of air collected by Mr. WELSH in the course of the balloon ascents made under the superintendence of the Kew Committee of the British Association, may not be unacceptable to the Fellows of the Royal Society as supplementary to a part of Mr. WELSH's report and observations.

"The samples of air collected upon the 26th of August appear to have been taken in the most unexceptionable manner, and it was upon these only that my experiments were made. The recipients for the air were wide glass tubes, about 5 cubic inches in capacity, to each of which a portion of barometric tubing, 3 or 4 inches in length, was attached, as a neck that might receive a cap and stopcock, and which would admit also of being hermetically sealed afterwards by the blowpipe. Two of these tubes were furnished with excellent stopcocks, and were found able to support without leakage for twenty-four hours the exhaustion obtained by an air-pump, the gauge of which indicated a pressure of 0·5 inch.

"Having been thus tested they were exhausted to this extent immediately before the ascent took place, and were filled with the specimens to be examined by simply opening and then closing the stopcock, the altitude being determined by an observation of the barometer at the moment. In the third tube, a Torricellian vacuum was obtained, the tube being then sealed and drawn off, so as to admit of being broken at a filemark when the air was to be collected; after the specimen had been thus obtained, the aperture was closed by thrusting the neck of the tube into a cap filled with softened wax.

"The tubes were within twenty hours after the air had been collected hermetically sealed by myself, and the proportions of oxygen and nitrogen determined with great care by detonation with hydrogen in 'REGNAULT'S Eudiometer.'

"The volumes of oxygen found in the air collected at different altitudes are given in the following table:—

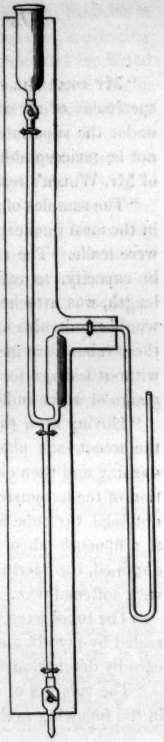
	Altitude.	Volume of oxygen.
Air collected at King's College		20·920
Tube 2	13,460 feet	20·888
Tube 3	18,000 feet	20·747
Tube (G 1), Torricellian vacuum	18,630 feet	20·888

"From these observations it would appear that the composition of the atmosphere, as regards the proportion of oxygen and nitrogen, scarcely varies more as we ascend through the first half of that atmosphere (for at an altitude of about 3½ miles one-half of the atmosphere lies beneath us), than it is found to vary at different spots upon the surface: that there is, in fact (as GAY-LUSSAC had long since announced as the result of his experiments, made at a time when the methods of gaseous analysis were

less perfect than at present), no sensible difference in the composition of the atmosphere upon the surface, and at the greatest heights accessible to man.

"In quantities of air so limited as those at my disposal, it was not possible to determine accurately the proportion of carbonic acid which they contained. Its presence however was distinctly shown by the formation of a film of carbonate of lead upon a solution of the subacetate which was introduced to a portion of the air confined over mercury.

"I have found a form of pipette, a sketch of which I subjoin, very useful for transferring small quantities of gases over mercury. It saves a great deal of fatigue, and I think contributes to precision in the results obtained. Its working is so simple as hardly to require description. It is first completely filled with mercury by closing the lower steel stopcock and opening the upper one, then pouring in mercury by the funnel until the metal escapes by the open end of the long bent tube; the upper stopcock is now closed, the bent tube introduced into the jar containing the gas to be transferred, and the end of the tube is lifted above the level of the metal in the jar of gas; the lower stopcock is then opened, mercury runs out, and gas takes its place; when a sufficient quantity has entered, the end of the tube is depressed beneath the mercury, a little of the metal enters and seals the opening, the lower stopcock is closed, and the pipette with its contents is withdrawn: the bent tube is now introduced beneath the jar which is to receive the gas. The funnel at top is filled with mercury, the upper stopcock opened, and the descending column of mercury expels the gas into the vessel destined to receive it.



"I am, my dear Sir,

"Yours very truly,

"WILLIAM ALLEN MILLER."

"*To Colonel Sykes,*
Chairman of the Kew Committee."

Meteorological Observations during the Ascents at different places.

Table III.—Places at which Meteorological Observations were taken in connection with the Balloon Ascents of 1852, with the Geographical coordinates, and name of the Observer or Authority.

Name of Place.	Latitude.	Longitude.	Height above sea level.	Authority.
	° ′ ″	° ′ ″		
Armagh	54 21 13	6 38 52 W.	209	Rev. Dr. Robinson, F.R.S.
Aylesbury	51 49	0 49 15 W.	284	Thomas Dell, Esq.
Bedford	52 8	0 28 W.	100	Dr. S. Herbert Barker.
Cambridge	52 12 52	0 5 53 E.	80	Professor Challis, F.R.S.
Cardington	52 7	0 24 W.	.....	S. C. Whitbread, Esq.
Derby*	52 55	1 28 16 W.	100?	Mr. Davis, Optician.
Diss*	52 22	1 6 E.	130	Thomas E. Amyot, Esq.
Dublin	53 21	6 15 W.	24½	Rev. R. V. Dixon, A.M.
Edinburgh	55 27 23	3 10 45 W.	354	Professor Smyth, F.R.S.E.
Enfield	51 39	0 4 57 W.	76½	Rev. J. M. Heath.
Grantham	52 54 52	0 39 0 W.	190	J. W. Jeans, Esq.
Greenwich	51 28 38	0 0 0	159	The Astronomer Royal.
Hartwell House	51 49	0 51 W.	250	Dr. Lee, F.R.S.
Hartwell Rectory	51 48 36	0 51 W.	290	Rev. C. Lowndes, M.A., F.R.A.S.
Haverhill	52 5	0 26 36 E.	.....	Wm. W. Boreham, Esq., F.R.A.S.
Hawerden	53 11 0	3 2 0 W.	260	Dr. T. Moffat, F.R.A.S.
Highfield House	52 57 30	1 10 W.	204	E. J. Lowe, Esq., F.R.A.S.
Holkham	52 57 10	0 48 E.	39	Samuel Shellabear, Esq.
Kew Observatory	51 28 37	0 15 45 W.	40?	Captain Younghusband, R.A., F.R.S.
Lewisham	51 28	0 1 W.	80	James Glaisher, Esq., F.R.S.
Linslade	51 55	0 40 W.	313	John Osborn, Jun., Esq.
Marboué	48 6 57	1 20 3 E.	361	M. Le Commandant Delcros.
Norwich	52 37	1 16 E.	33	Wm. Brooke, Esq.
Oxford	51 45 40	1 15 30 W.	210	M. J. Johnson, Esq., F.R.S.
Rosehill, Oxford	51 43 50	1 14 W.	270?	John Slatter, Esq.
Les Rousseaux	47 30 58	2 20 10 E.	.....	M. Le Commandant Delcros.
Royston	52 2 40	0 0 30 W.	271	Hale Wortham, Esq.
Ryde	50 45	1 11 30 W.	110	Benjamin Barrow, Esq.
Southampton	50 54 34	1 24 25 W.	60	Dr. Drew, F.R.A.S.
Stone	51 47 57	0 52 16 W.	320	Rev. J. B. Reade, M.A., F.R.S.
St. Ives*	52 20	0 5 W.	.....	John King Watts, Esq.
St. John's Wood	51 31	0 15 W.	150	George Leach, Esq.
Ventnor*	50 36	1 13 W.	150	Dr. Martin.
York	53 57 48	1 4 W.	50	John Ford, Esq.

* The barometrical observations at Derby, Diss, St. Ives, and Ventnor have not been corrected for temperature.

TABLE IV.—Meteorological Observations, made at Various Places on the days of the Four Balloon Ascents in 1852.

Observations on August 17, 1852.

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
h	in.	°	°	in.	in.	°	°	in.	in.	°	°	in.	in.	°	°	in.
	Aylesbury.				Bedford.				Cambridge.				Cardington.			
9 A.M.	.....	...	...	.....	29·79	67·4	62·0	0·498	29·912	65·5	63·0	0·550	29·793	65·0	60·5	0·482
2 P.M.	29·568	72·3	66·7	0·590												
3	·523	73·0	68·5	·642	.....	...	...	.....	·813	72·0	68·6	·656	·725	70·8	65·0	·552
4	·515	69·8	65·7	·585	·70	73·1	67·0	·590	·814	71·3	68·5	·660	·717	70·0	66·0	·593
5	·502	69·3	66·0	·601	·66	70·6	66·8	·611	·789	70·3	68·1	·658	·669	69·2	66·0	·602
6	·485	67·0	65·7	·616	·65	69·6	67·0	·630	·774	69·3	67·7	·656	·678	67·4	66·0	·622
7	·477	66·0	65·0	·606	·64	68·4	66·5	·627	·756	67·8	66·5	·622	·668	67·0	65·0	·595
8	·487	62·0	62·0	·559	·62	66·1	65·0	·605	·730	67·5	66·3	·631	·623			
	Derby.				Dublin.				Diss.				Enfield.			
2 P.M.	29·74	71·0	64·0	0·518	.....	...	...	.....	29·85	70·0	66·0	0·593				
3	·74	72·0	65·0	·538	29·679	69·2	62·7	0·499	·80	70·0	66·0	·593	29·754	72·8	66·0	0·561
4	·71	70·0	64·5	·545	·682	68·5	62·0	·486	·78	69·5	66·0	·599	·735	71·7	66·6	·593
5	·71	68·0	65·0	·583	·682	67·6	61·8	·490	·78	69·0	66·0	·604	·707	69·9	66·0	·594
6	·69	67·0	64·0	·563	·686	66·9	60·9	·472	·76	68·5	66·5	·626	·696	69·5	66·0	·599
7	·68	65·5	63·5	·565	·689	63·5	59·0	·456	·74	67·0	65·5	·610	·657	68·9	66·0	·605
8	·68	64·5	63·0	·561	·709	62·5	58·9	·464	·73	65·0	64·0	·586	·650	66·8	64·8	·591
	Edinburgh.				Haverhill.				Hawerden, Chester.				Highfield House.			
9 A.M.	.....	...	...	.....	.....	...	...	.....	.....	...	...	.....	29·722	69·2	63·3	0·517
2 P.M.	.....	...	...	.....	29·750	72·0	65·5	0·554	29·450	67·5	63·0	0·528				
3	29·279	67·7	...	.....	·734	72·0	66·3	·580	·454	67·8	64·0	·554	·653	75·0	67·0	·569
4	·264	67·5	...	.....	·720	73·0	67·3	·601	·430	68·0	64·0	·552	·652	71·5	65·3	·553
5	·282	66·2	...	.....	·705	71·0	67·5	·631	·428	66·9	63·9	·561	·639	68·8	65·2	·581
6	.....	...	...	.....	·674	69·8	66·8	·588	·420	66·5	64·0	·569	·623	67·0	64·5	·579
7	.....	...	...	.....	·663	68·0	66·5	·631	·420	65·8	63·5	·561	·607	65·0	63·9	·583
8	.....	...	...	.....	·629	66·2	64·8	·597	·414	...	...	.....	·594	64·2	63·0	·565
	Holkham.				Linslade.				Norwich.				Oxford.			
9 A.M.	29·888	66·5	62·4	0·520	29·595	64·0	59·3	0·558	29·906	68·0	64·0	·552				
2 P.M.	.....	...	...	.....	·525	69·3	63·3	·516	.....	...	...	.....	29·630	72·1	66·4	0·583
3	·796	74·4	67·2	·582	·505	71·3	64·2	·521	·786	73·0	68·0	·625	·615	70·4	66·0	·589
4	·785	72·3	66·4	·581	.....	70·2	64·8	·553	·766	72·0	68·0	·636	·577	67·6	65·1	·591
5	·784	72·2	67·0	·601	·457	68·5	64·9	·575	·756	72·0	68·0	·636	·587	67·7	65·0	·587
6	·763	68·8	65·2	·581	·454	67·4	65·1	·593	·748	70·0	67·0	·625	·507	66·0	64·8	·600
7	·756	67·2	64·7	·583	.....	...	...	.....	·716	68·0	66·0	·616	·528	65·8	64·8	·602
8	·735	66·3	64·0	·571	·420	63·9	62·3	·546	·709	65·0	64·5	·601	·512	65·1	64·7	·607
	Rosehill, Oxford.				Ryde.				Southampton.				St. Ives.			
9 A.M.	29·639	65·0	61·0	0·497	29·850	63·8	61·9	0·536	29·812	66·1	63·0	0·544				
2 P.M.	.....	...	...	.....	·748	65·9	65·7	·629	·729	71·0	67·0	·615	29·70	73·0		
3	·554	68·6	64·7	·568	·731	66·8	66·2	·635	·711	67·7	66·0	·619	·69	72·0		
4	·507	66·5	64·5	·585	·667	65·8	64·2	·583	·674	67·9	65·6	·603	·68	72·0		
5	·529	65·0	64·8	·611	·667	65·8	63·7	·568	·656	67·0	66·0	·627	·68	72·0		
6	·443	65·0	64·6	·605	·639	64·9	64·2	·593	·626	66·0	65·0	·606	·65	70·0		
7	.....	...	...	.....	·622	64·8	63·7	·579	·624	64·0	62·9	·574	·65	68·0		
8	.....	...	...	.....	·623	64·0	63·5	·581	(?) ·867	63·5	62·0	·542	·58	67·0		

Table IV. (Continued.)

Observations on August 17, 1852.

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
h	in.	°	°	in.	in.	°	°	in.	in.	°	°	in.	in.	°	°	in.
	St. John's Wood.				York.											
9 A.M.	29·785	62·8	60·3	0·500	29·721	66·0	63·0	0·545								
2 P.M.																
3	·663	69·8	66·3	·605	·734	71·0	64·5	·535								
4	·647	69·5	66·5	·615	·710	71·0	64·0	·519								
5	·608	68·8	65·8	·600	·701	69·0	64·0	·542								
6	·596	68·6	66·2	·615	·701	69·0	64·0	·542								
7	·552	68·8	66·4	·620	·713	68·0	63·5	·537								
8	·580	65·0	64·0	·586	·707	66·0	64·0	·575								
	Armagh.				Grantham.				Greenwich.				Lewisham.			
9 A.M.	29·297	65·5	62·6	0·538	29·688	66·4	61·4	0.493	29·804	64·0	58·1	0·426				
3 P.M.	.....	...	...	.....	·623	70·3	64·0	·527	·692	75·4	65·6	·520	29·762	74·9	69·2	0·645
3½	.....	...	...	.....	·615	69·6	63·6	·522	·689	74·0	66·1	·552	·755	72·3	67·0	·600
4	.....	...	...	.....	·617	69·0	63·6	·529	·683	71·7	65·0	·543	·751	71·7	67·7	·630
4½	.....	...	...	.....	·618	68·5	63·7	·538	·655	71·1	64·8	·543	·725	71·6	67·4	·621
5	·437	64·2	58·6	·437	·600	68·0	63·5	·537	·636	70·6	65·0	·555	·714	70·7	67·5	·635
5½	·440	64·6	58·8	·438	·571	67·5	63·5	·544	·623	69·8	65·0	·564	·686	70·0	67·1	·629
6	·449	65·0	59·1	·442	·579	67·4	63·5	·544	·621	69·1	65·1	·575	·688	69·1	67·0	·636
6½	·446	64·5	59·1	·442	·584	66·9	63·1	·538	·619	69·1	65·1	·575	·678	68·8	66·9	·636
7	·446	64·1	58·6	·438	·561	66·2	62·8	·536	·602	68·6	65·4	·590	·648	68·4	67·6	·652
7½	·464	61·4	57·2	·429	·551	65·9	62·6	·533	·607	61·4	60·5	·521	·656	64·0	63·7	·588
8	·451	59·6	57·0	·444	·546	65·6	62·6	·537	·593	62·5	61·0	·524	·672	63·2	62·8	·570
8½	.....	...	...	.....	.....	...	...	.....	·593	63·5	61·6	·530				
9	.....	...	...	.....	·548	64·8	63·0	·558	·593	64·2	62·9	·562				

Observations on August 26, 1852.

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
	Aylesbury.				Bedford.				Cambridge.				Derby.			
9 A.M.	.....	...	...	.....	30·030	62·6	58·8	0·459	30·023	64·2	60·0	0·476				
2 P.M.	29·721	72·3	66·0	0·568												
3	·722	73·5	66·3	·564	·043	72·1	65·8	·562	·020	68·4	64·9	·576	29·97	68·0	62·0	0·492
4	·727	70·6	65·0	·555	·062	70·0	65·0	·561	·017	67·8	64·8	·579	·98	68·0	63·0	·522
5	·734	69·9	65·4	·575	·070	69·4	64·5	·552	·018	67·7	64·1	·559	·98	67·0	62·0	·503
6	·744	66·0	63·3	·553	·070	66·5	63·0	·539	·031	66·4	63·0	·540	29·98	65·0	61·0	·496
7	·759	60·7	59·7	·507	·079	64·0	61·5	·522	·056	63·2	60·9	·513	30·00	62·0	59·0	·472
8	·775	58·7	58·2	·486	·079	61·2	60·0	·510	·065	61·0	58·8	·477	30·00			
	Diss.				Edinburgh.				Enfield.				Holkham.			
9 A.M.	.....	...	...	.....	.....	...	...	.....	29·916	63·0	59·5	0·475	30·004	58·6	57·4	0·465
2 P.M.	29·97	64·0	60·0	0·478												
3	·97	65·5	61·0	·491	.....	...	...	.....	·941	70·9	64·6	·538	·037	67·0	60·8	·468
4	·97	65·0	61·0	·496	.....	...	...	.....	·942	69·8	63·9	·529	·038	66·0	59·2	·433
5	·98	63·5	61·5	·528	29·634	66·7	...	.....	·945	69·2	63·4	·520	·037	63·6	58·1	·430
6	·98	62·5	59·5	·480	.....	...	...	.....	·955	67·8	63·0	·524	·055	61·7	58·0	·448
7	·99	59·5	57·0	·445	.....	...	...	.....	.....	...	...	.....	·064	60·4	57·8	·457
8	.....	...	...	.....	.....	...	...	.....	.....	...	...	.....	·080	59·8	57·2	·447
	Norwich.				Oxford.				Royston.				Southampton.			
9 A.M.	.....	...	...	.....	.....	...	...	.....	.....	...	...	.....	29·911	64·1	62·4	0·547
2 P.M.	30·039	66·0	60·0	0·456	29·820	72·1	65·1	0·541								
3	·040	66·0	60·0	·456	·822	71·0	64·6	·538	29·784	70·1	63·3	0·507				
4	·044	65·5	60·0	·461	·820	70·1	63·3	·507	·789	69·4	63·2	·512				
5	·046	65·0	60·0	·467	·832	68·4	64·0	·548	·799	66·8	62·2	·512				
6	·052	62·0	59·0	·472	·830	67·0	63·8	·557	·798	63·7	60·0	·482	·925	69·0	66·5	·620
7	·058	59·0	57·5	·464	·848	65·3	62·2	·528	·822	60·3	58·5	·477	·929	65·0	63·7	·576
8	·066	55·0	54·5	·429	·857	63·0	61·0	·519	·827	58·0	56·8	·457				

TABLE IV. (Continued.)

Observations on August 26, 1852.

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
h	in.	°	°	in.	in.	°	°	in.	in.	°	°	in.	in.	°	°	in.
	Stone.				St. Ives.				St. John's Wood.							
9 A.M.	29·637	67·4	64·0	0·559	.....	...	...	.....	29·862	62·8	61·3	0·529				
2 P.M.	·646	72·6	65·6	·551	29·85	72·0	...	.....	.....	...	...	.....				
3	·648	68·9	64·0	·543	·85	71·0	...	.....	·866	68·5	64·2	·553				
4	·646	68·9	64·5	·558	·85	69·0	...	.....	·874	67·8	63·8	·548				
5	·656	68·7	63·5	·529	·86	69·0	...	.....	·879	66·8	63·8	·560				
6	·664	66·6	62·9	·535	·87	68·0	...	.....	·888	64·8	62·8	·552				
7	·689	63·0	60·3	·498	·89	64·0	...	.....	·911	62·8	60·4	·503				
8	·694	60·9	58·8	·479	·90	61·0	...	.....	·920	61·8	59·8	·497				
	Grantham.				Greenwich.				Kew.				Lewisham.			
9 A.M.	29·850	61·8	57·4	0·430	29·873	66·8	62·2	0·512								
Noon	.....	...	...	.....	·879	69·6	63·6	·522								
3 P.M.	·869	64·5	58·7	·436	·880	71·1	64·1	·521	.....	...	...	.....	29·958	72·7	66·5	0·579
3½	·872	65·0	59·3	·447	·880	71·1	64·1	·521	29·978	69·6	64·4	0·547	·952	72·5	66·2	·571
4	·873	65·4	59·7	·454	·876	70·9	63·8	·514	·992	69·1	64·0	·540	·956	71·8	66·0	·573
4½	·872	65·9	60·2	·462	·888	69·6	63·1	·507	·995	68·2	63·9	·547	·968	68·3	64·0	·549
5	·875	65·4	60·0	·463	·893	67·1	61·6	·489	·993	67·2	63·1	·534	·970	67·8	63·7	·545
5½	·879	64·5	58·5	·430	·887	66·6	62·0	·507	·992	66·7	63·0	·537	·967	67·5	63·5	·542
6	·883	64·0	57·9	·419	·890	64·5	60·2	·478	29·996	65·1	61·4	·507	·982	65·5	62·6	·538
6½	·884	63·0	57·7	·425	·897	63·8	59·6	·469	30·006	63·5	60·9	·510	·983	64·0	61·6	·524
6½	.....	...	...	.....	.....	...	...	.....	·019	63·1	60·5	·503				
7	·885	61·0	57·3	·437	·911	62·7	59·1	·466	.....	...	...	.....	·989	62·7	61·0	·522
7½	·893	60·6	56·7	·424	·913	62·0	58·8	·466	.....	...	...	.....	29·994	62·2	60·5	·513
8	·897	59·6	56·3	·425	.....	...	...	.....	.....	...	...	.....	30·001	61·5	60·0	·506
8½	.....	...	...	.....	.....	...	...	.....	.....	...	...	.....	·009	60·6	59·2	·493
9	·903	57·9	55·6	·425	·929	59·3	57·3	·456	.....	...	...	.....	·013	59·6	58·5	·485

Observations on October 21, 1852.

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
	Bedford.				Enfield.				Hartwell House.				Hartwell Rectory.			
9 A.M.	29·94	46·6	46·0	0·318	29·984	46·5	45·9	0·317	29·810	49·3	47·8	0·329	29·695	47·8	47·8	0·346
1 P.M.	·88	56·6	53·5	·385	·917	52·0	50·2	·356								
2	·86	57·0	54·0	·394	·906	52·0	50·9	·373	·766	56·8	51·0	·321				
3	·85	56·4	53·0	·375	·891	52·0	51·5	·387	·766	55·8	51·0	·332				
4	·85	55·0	51·8	·361	·871	52·1	51·8	·394	·750	55·3	51·2	·343				
5	·85	54·0	51·7	·369	·871	52·0	52·0	·400	·742	54·0	51·0	·352				
6	·87	54·0	51·7	·369	·886	51·5	51·0	·380	·754	54·9	51·9	·364	29·626	54·0	52·5	·389
7	·87	54·1	52·0	·376	.....	...	...	.....	·734	54·6	53·0	·395				
	Linslade.				Norwich.				Oxford.				Rosehill, Oxford.			
9 A.M.	29·728	48·1	47·5	0·336	.....	...	...	.....	.....	...	...	.....	29·759	48·4	48·3	0·351
1 P.M.	·693	54·5	51·3	·354	.....	...	...	.....	29·789	55·0	52·8	0·386	·720	53·5	51·8	·378
2	·665	55·2	50·5	·327	29·932	53·0	51·0	0·364	·763	54·9	52·8	·387	·705	53·4	51·4	·369
3	·659	55·0	50·9	·339	·932	53·0	51·0	·364	·758	54·7	52·3	·377	·694	53·8	52·0	·380
4	·645	53·9	50·5	·342	·916	52·5	51·4	·379	·736	54·1	52·5	·388	·672	53·8	52·2	·384
5	·639	53·1	50·1	·340	·906	51·0	50·0	·362	·736	54·0	52·6	·392	·677	53·1	52·1	·390
6	·639	52·6	50·1	·346	·906	48·0	47·0	·326	·736	54·0	53·0	·402				
7	·623	51·9	51·2	·381	·906	47·8	47·7	·344	·731	54·0	53·2	·407				
	Ryde.				Stone.				St. John's Wood.				Ventnor.			
9 A.M.	29·933	53·1	52·0	0·388	29·649	48·3	47·8	0·340	29·892	46·3	45·6	0·313				
1 P.M.	·897	59·8	55·5	·402	·617	54·6	52·4	·380	·827	56·0	52·5	·367	29·912	60·0	56·0	0·412
2	.....	...	...	.....	·597	56·3	53·5	·389	·815	55·8	53·3	·390	·892	60·0	57·0	·440
3	·889	59·0	55·0	·397	·586	54·4	51·6	·362	·810	57·3	53·8	·386	·874	60·0	57·0	·440
4	.....	...	...	.....	·574	53·7	51·6	·370	·795	54·5	51·9	·369	·872	60·0	57·0	·440
5	.....	...	...	.....	·571	53·0	51·3	·371	·795	54·9	52·8	·387	·872	60·0	57·0	·440
6	·887	56·0	54·8	·426	·578	53·4	51·5	·372	·804	54·8	52·8	·389	·874	60·0	57·0	·440
7	.....	...	...	.....	·562	52·7	51·5	·379	·792	54·0	52·5	·389	·874	60·0	57·0	·440
8	.....	...	...	.....	·554	52·7	51·9	·389	.....	...	...	.....				
9	.....	...	...	.....	·550	52·7	51·9	·389	·783	53·0	52·0	·389				

Table IV. (Continued.)

Observations on October 21, 1852.

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
h	in.	°	°	in.
Grantham.				
9 A.M.	29·851	47·2	46·4	0·321
Noon	.....	...	...	.....
1 P.M.	·794	51·6	49·5	·343
1½	.....	...	...	.....
2	·786	53·0	50·4	·349
2½	.....	...	...	.....
3	·760	52·5	49·4	·331
3½	.....	...	...	.....
4	·768	51·7	49·8	·350
4½	.....	...	...	.....
5	·747	51·0	49·6	·352
5½	.....	...	...	.....
6	·748	51·0	49·5	·350
6½	.....	...	...	.....
7	·743	51·0	49·9	·360
9	·717	51·6	50·3	·362
Greenwich.				
9 A.M.	29·887	48·3	46·0	0·299
Noon	·860	58·2	53·4	·365
1 P.M.	·845	59·7	53·7	·356
1½	.....	...	...	.....
2	·838	57·9	51·6	·323
2½	·836	58·7	51·9	·322
3	·835	58·7	52·4	·334
3½	·832	57·4	52·4	·349
4	·828	56·9	52·1	·347
4½	·826	56·4	51·9	·348
5	·831	55·9	51·9	·353
5½	·826	55·4	51·9	·359
6	·813	55·9	51·9	·353
6½	·806	54·8	51·4	·353
7	.....	54·2	51·0	·350
9	·796	52·8	50·0	·341
Lewisham.				
9 A.M.	29·958	45·5	45·0	0·308
Noon				
1 P.M.	·907	59·1	53·8	·366
1½	·907	58·6	52·8	·346
2	·908	58·0	52·0	·333
2½	·908	58·6	52·1	·328
3	·907	58·7	53·7	·367
3½	·901	58·0	53·6	·373
4	·893	56·9	54·1	·397
4½	·893	55·7	54·0	·409
5	·891	54·7	53·3	·402
5½	·892	54·2	53·2	·405
6	·891	55·0	52·2	·370
6½	·895	55·3	53·0	·387
7	·892	53·0	52·0	·389
9				

Observations on November 10, 1852.

Hour.	Barom.	Therm. Dry.	Therm. Wet.	Tension of Vapour.
Bedford.				
9 A.M.	29·97	46·6	43·6	0·265
3 P.M.	·96	47·6	44·0	·263
Cambridge.				
1 P.M.	30·061	47·8	44·2	0·265
2	·058	48·2	44·0	·256
3	·056	47·1	43·8	·264
4	·047	46·5	44·0	·275
5	·033	46·1	43·5	·268
6	·029	44·6	43·5	·285
7	·025	44·2	43·5	·290
Diss.				
10 A.M.	29·99	46·0	44·5	0·292
1½ P.M.	·98	48·5	46·0	·296
2½	·98	48·0	45·0	·280
3½	·98	47·5	44·5	·275
5½	·96	44·5	43·5	·287
7	·96	43·5	43·0	·287
Enfield.				
9 A.M.	29·991	49·8	46·8	0·300
6 P.M.	·940	...	...	.....
Grantham.				
9 A.M.	29·922	41·6	39·5	0·236
1 P.M.	·904	44·7	42·5	·263
2	·893	44·4	42·3	·262
3	·884	44·3	42·3	·263
4	·880	43·7	42·0	·264
5	·869	43·2	41·8	·265
6	·871	42·4	41·2	·262
7	·860	42·2	40·8	·256
9	·834	41·5	40·4	·256
Greenwich.				
9 A.M.	29·937	48·6	45·9	0·293
Noon	·928	51·0	44·6	·237
1 P.M.	·918	50·5	45·7	·267
1½	·920	50·5	45·7	·267
2 P.M.	29·918	50·1	45·4	0·265
2½	·917	48·6	45·0	·273
3	·914	48·0	44·5	·258
3½	·911	49·0	44·7	·262
4	·910	48·9	44·6	·261
4½	·910	47·7	44·0	·262
5	·905	47·3	43·6	·257
5½	·893	46·3	42·5	·244
6	·893	46·1	42·6	·249
6½	·900	46·1	42·5	·247
7	·894	46·1	42·6	·249
Hartwell House.				
9 A.M.	29·756	47·0	45·8	0·309
3 P.M.	·712	48·5	44·5	·263
Lewisham.				
9 A.M.	30·015	49·2	47·6	0·326
1 P.M.	·005	51·0	47·8	·309
1½	30·001	51·0	47·6	·305
2	29·987	50·8	47·5	·305
2½	·984	50·7	47·6	·308
3	·980	49·3	46·3	·295
3½	·975	49·6	46·5	·295
4	·975	49·0	45·7	·284
4½	·974	47·9	45·4	·291
5	·974	47·7	45·5	·295
5½	·966	47·0	44·8	·287
6	·964	47·0	44·7	·285
6½	·962	46·8	44·5	·283
7	·951	46·6	44·2	·279
Norwich.				
9 A.M.	30·013	46·7	43·7	0·266
Noon	29·997	48·4	43·4	·240
3 P.M.	·993	48·3	42·3	·218
6	.....	44·7	42·4	·261
9	.....	43·7	42·7	·279
Oxford.				
10 A.M.	29·891	48·3	45·4	0·286
1 P.M.	·863	49·5	46·2	·291
2	·859	49·0	46·0	·291
3	·852	48·9	45·3	·277
4	·836	48·1	45·3	·286
5½	·832	47·0	44·1	·272
6	·836	46·9	44·0	·271
7	·821	46·1	43·5	·269
Rosehill, Oxford.				
9 A.M.	29·829	47·2	44·8	0·285
Royston.				
10 A.M.	29·836	47·5	44·7	0·279
6 P.M.	·789	43·5	42·4	·275
Ryde.				
9 A.M.	30·023	51·6	50·5	0·368
Southampton.				
1½ P.M.	29·966	52·6	50·0	0·343
2	·959	52·9	50·0	·340
3	·942	52·9	50·3	·347
4	·945	51·5	50·0	·356
6	·938	50·6	49·6	·357
Stone.				
9 A.M.	29·707	46·3	44·2	0·282
9 P.M.	·627	44·7	42·6	·265
St. Ives.				
1 P.M.	29·44	44·0		
2	·42	44·0		
3	·37	45·0		
4	·35	48·0		
5	·28	48·0		
6	·27	48·0		
7	·27	48·0		
St. John's Wood.				
9 A.M.	29·932	48·3	46·3	0·306
5 P.M.	·891	46·6	44·1	·277
6	·884	46·3	43·6	·268

TABLE IV. (Continued.)

Hour.	Barom.	Therm.		Tension of Vapour.	Barom.	Therm.		Tension of Vapour.	Barom.	Therm.		Tension of Vapour.	Barom.	Therm.		Tension of Vapour.
h	in.	Dry. °	Wet. °	in.	in.	Dry. °	Wet. °	in.	in.	Dry. °	Wet. °	in.	in.	Dry. °	Wet. °	in.
	Les Rousseaux, Aug. 17.				Les Rousseaux, Aug. 26.				Marboué, Oct. 21.				Marboué, Nov. 10.			
7 A.M.	29·475	67·1	62·0	0·502	29·443	63·1	59·0	0·460	29·752	34·9	32·7	0·179	29·719	43·0	41·2	0·255
9	·456	69·1	63·1	·513	·445	72·0	61·3	·427	·752	52·0	47·1	·283	·703	49·3	46·4	·297
Noon	·395	76·7	66·4	·534	·449	78·8	66·9	·525	·691	60·1	53·2	·338	·668	57·6	52·7	·354
3 P.M.	·296	80·8	68·2	·546	·454	78·3	67·6	·553	·667	59·2	54·1	·371	·650	56·3	50·7	·319
9	·297	70·5	67·3	·630	·494	67·8	63·0	·524	·635	49·5	46·8	·304	·628	44·6	41·9	·252

DESCRIPTION OF THE PLATES.

PLATE XIX. XX. XXI. and XXII.

The results for each ascent of the observations of temperature, tension of vapour, and relative humidity are projected in these Plates. For the most part each individual observation is given, except when they were very numerous and occurring at too close intervals of height to be easily represented. In such cases groups have been taken, but no group ever contains more observations than were recorded within 200 feet.

The ordinates represent the height above the level of the sea, one division being equivalent to 200 feet; the abscissæ representing the temperature of the air, the tension of vapour, or the relative humidity.

The scale employed for the temperature is one division to 2° FAHR.; for the tension of vapour, ten divisions to 0·2 inch of pressure of mercury; and for the relative humidity, twenty divisions to the whole range 0—100.

The straight lines drawn through the curves of temperature are deduced from the results of equation (1.) for the upper and lower divisions in each series (see p. 25).

The points ⊙ in the curves of tension of vapour are from the indications of REGNAULT'S hygrometer.

In the divisions occupied by the relative humidity the strong vertical lines correspond to the heights at which clouds existed in the air, dotted lines being drawn when the cloud was only partial.

XIII. *On certain Functions of the Spinal Chord, with further Investigations into its Structure.* By J. Lockhart Clarke, *Esq.* *Communicated by* Samuel Solly, *Esq., F.R.S.*

Received April 6,—Read April 14, 1853.

WHEN I had the honour of laying before the Royal Society my former researches on the structure of the spinal chord, I intimated an intention of preparing another communication on the structure of the medulla oblongata and cerebellum; but as many important points in the minute anatomy of the chord still remained in obscurity, I thought it advisable to make them first the subject of special inquiry, as far as the new method I employed would enable me to proceed. Moreover, as all investigations into the structure of any organ have, or ought to have, for their object a clearer and better knowledge of its functions, I have undertaken also to communicate in this paper whatever physiological deductions may appear to follow from my observations. Having no particular theory to support, and being influenced in these inquiries by no other feeling than the simple desire to elicit truth, the greatest care has been taken to verify my facts, and caution has been exercised in drawing conclusions from them.

It is a question of great interest and physiological importance, whether the roots of the spinal nerves belong exclusively to the spinal chord, or whether part of them ascend within either the white or the grey columns, and form the channels by which impressions are transmitted to and from the brain. On account of its interest and importance, I have employed much time and labour in endeavouring to arrive at some well-grounded and settled conclusion on this very difficult subject, having devoted to it alone many hours daily for nearly five months. So extremely intricate, however, is the internal structure of the chord; so numerous are the planes in which the nerve-roots enter the grey substance; and so various are the directions which they pursue within it, that notwithstanding the perfect transparency of my preparations, and the sharp outline which their fibres retain, my efforts to determine the exact relation between these roots and the white and grey columns appeared for some time almost hopeless; but by varying my dissections according to the exigencies of each case of difficulty, I succeeded in arriving at several results which I believe will be considered important.

That part of the fibres composing the anterior and posterior roots of the spinal nerves ascend longitudinally with the white columns, without entering the grey substance of the chord, and transmit to and from the brain impressions which give rise

to sensation and voluntary motion, is the opinion maintained by many eminent physiologists and excellent observers.

In support of this doctrine, Mr. Grainger*, Mr. Solly†, and Dr. J. Budge‡ have adduced the fact of having *traced* these fibres in the spinal chord of the vertebrata. A similar fact is said to have been established by Mr. Newport§ and Dr. Carpenter in the invertebrata. On the other hand, the opinion that all the fibres of the spinal nerves enter the grey substance and belong exclusively to the spinal chord, is held by Todd and Bowman‖, Stilling¶, Volkmann** and others.

In making the preparations for the investigation of this subject, I employed the second method described in my former communication on the Spinal Chord††. The animals selected for this purpose were the Ox, Calf, Cat, Rat, Mouse, and Frog. After many fruitless attempts I succeeded in rendering perfectly transparent the entire chords of the Mouse and Frog, but found that their cylindrical form interfered considerably with a distinct view of their several parts, and rendered the examination unsatisfactory. Having succeeded, however, in a similar way, with longitudinal sections of the spinal chord of the Cat, of at least $\frac{1}{12}$th of an inch in thickness and 2 inches in length, with the anterior and posterior roots attached, I was delighted, on examining these sections, with a view of its internal structure which far surpassed any I had hitherto been able to obtain.

On a former occasion, I showed unequivocally, that to the posterior white columns the posterior roots, and to the anterior white columns the anterior roots, of the spinal nerves are exclusively attached; while the lateral columns to which both these roots were formerly supposed to be connected, are in immediate connection only with the spinal-accessory nerve‡‡.

Of the Posterior Roots.—Plate XXIII. exactly represents a longitudinal section through the cervical enlargement of the spinal chord of the Cat, from the eighth to the twelfth pair of nerves. In this section the bundles which form the posterior roots P, P, P, P, are observed to consist of three kinds, which differ from each other partly in direction, and partly in the size of their component filaments.

The first kind, *a*, *a*, *a*, *a*, enter the chord transversely, and pursue a very remarkable course. I have not seen them distinctly below the cervical enlargement. Each bundle, after traversing the longitudinal fibres of the posterior columns P, C, in a compact form, and at right angles, continues in the same direction to a considerable but variable depth within the grey substance G, dilating and again contracting, so as to assume a fusiform appearance. It then bends round upon itself, at a right or more obtuse angle, and running for a considerable distance in a longitudinal direction

* Spinal Chord. † Human Brain. ‡ Müller's Archiv, 1844. § Philosophical Transactions, 1843.
‖ Physiological Anatomy, vol. i. 1845. ¶ Untersuchungen über die Textur des Rückenmarks, 1842.
** Nervenphysiologie, in Wagner. †† Philosophical Transactions, 1851, p. 608, Part II.
‡‡ Many of my preparations are in the possession of the Microscopical Society of London, and the Royal College of Surgeons.

down the chord, sends forwards, at short intervals, into the anterior grey substance, a series of fibres like those issuing from the roots of plants. In this longitudinal course it is joined by corresponding fibres from bundles above and below it, which thus contribute to form a continuous band.

The fibres projecting from this band into the anterior grey substance have the following distribution. Part of them form loops with each other within the grey substance, particularly near its border; others extend directly into the anterior white columns A, C, and bending round both *upwards* and *downwards*, are seen sometimes to re-enter the grey substance and form with each other a series of loops, and sometimes to continue a longitudinal course within the anterior white columns, amongst the fibres of which they become lost. Whether the latter, also, ultimately form broader loops with corresponding fibres from the grey substance, it is impossible to ascertain. But even if those which *ascend* in the anterior columns are continued upwards to the brain, one can scarcely avoid inferring that those which *descend* re-enter the grey substance, either to form loops, or to become continuous with the fibres of the anterior roots, since the whole of the latter, as we shall presently see, proceed directly to the grey substance. Indeed, I have sometimes felt almost persuaded that a great number of the fibres of these posterior roots are directly continuous, in the grey substance, with those of the anterior roots; but I cannot make this statement with absolute certainty; and as the question is one of extreme difficulty, I shall hereafter endeavour to make it a subject of special attention.

The second kind of bundles which form the posterior roots *b*, *b*, *b*, Plates XXIII. and XXIV. traverse the posterior columns transversely, and with different degrees of obliquity from without inwards, extending nearly as far as the posterior median fissure: see also Philosophical Transactions, 1851, Plate XXIII. fig. 14. Their component filaments are finer than those of the other kind of bundles, measuring, in a recent state, about the $\frac{1}{1000}$th of an inch in diameter. They enter and pass through the posterior grey substance at various angles, and in compact bundles which decussate and interlace each other in the most complicated manner. Some of their fibres cross over to the opposite side, through the posterior commissure, behind the spinal canal; others extend into the posterior and lateral white columns; and the rest may be traced deeply into the anterior grey substance, where they separate in various directions, and are ultimately lost to view.

The bundles which compose the third kind of posterior roots enter the chord obliquely, *c*, *c*, *c*, Plates I. and II. A few of their fibres proceed near the surface both upwards and downwards, and pass out again with the roots above and below them. The rest cross the posterior white columns obliquely, and chiefly *upwards*, a small number only passing downwards. Interlacing at the same time with each other and the roots already described, they diverge, and for the most part reach the grey substance at points successively more distant from their entrance in proportion to the obliquity of their course; the remainder or most divergent taking a longitu-

dinal course with the fibres of the posterior white columns, amongst which they are lost. It is impossible to say whether any of these longitudinal fibres are continued as far as the brain, or whether they ultimately reach the grey substance of the chord. It is also extremely difficult to trace the other fibres of these roots after they have reached the posterior grey substance. In some of my finest preparations, however, they may be seen to interlace each other in a kind of network: see Plate XXIV. A large proportion diverge abruptly in various directions, so that in any section they are always divided. Many of them, both singly and in small bundles, may be observed to form loops by returning to the white columns.

Of the Anterior Roots.—The anterior roots of the spinal nerves, as I formerly described them, traverse the anterior part of the antero-lateral columns in distinct and nearly straight bundles, Plates XXIII. and XXIV. A, A, A, A. They form no interlacement with each other, like the posterior roots, until they reach the grey substance. Here their fibres diverge in every direction, like the expanded hairs of a brush. Some, near the margin, are easily seen to form loops with those of contiguous bundles; others run outwards to the lateral columns, and inwards to the anterior columns after decussating in the anterior commissure with corresponding fibres from the opposite side. A large number diverge equally *downwards* and *upwards* for some distance in the grey substance, while the remainder pass more deeply backwards and are lost. In no single instance have I seen any portion of these roots take a longitudinal course on directly entering the anterior white columns.

But besides the transverse bundles which form the anterior roots, a continuous system of exceedingly fine transverse fibres may be seen to issue from the anterior grey substance. They pass through, nearly all at right angles to, the anterior white columns, and disappear as they proceed towards the surface of the chord; but as many of them may be observed to turn round and take a longitudinal direction, it is probable that at the points where they disappear they all follow the same course. Within the grey substance they wind about and are gradually lost, mingling with the fibres of the anterior roots, and with those proceeding from the fine bundles of the posterior roots, which, perhaps, are continuous with them.

It may then, I think, be fairly laid down as a well-established fact, that nearly all, if not the whole of, the fibres composing the roots of the spinal nerves, after passing through the anterior and posterior white columns of the chord, proceed at once to its grey substance; and that if any of them ascend *directly* to the brain, it must be *those only* of the *posterior* roots which run longitudinally in the posterior columns.

That many excellent observers, with inferior means of observation, have arrived at a different conclusion on this extremely difficult subject, is not at all surprising. The opinions of Mr. Grainger are expressed in the following quotations from his excellent work on the spinal chord:—"After the *two* roots have perforated the theca vertebralis, and so reached the surface of the chord, it is well known that their fibres begin to separate from each other; of these fibres some are lost in the white

substance, whilst others, entering more deeply into the lateral furrows, are found to continue their course, nearly at right angles with the spinal chord itself, as far as the grey substance, in which they are lost. . . . . In examining the roots of the nerves, I have always relied on the assistance of the naked eye only, avoiding, for fear of deception, the use of a lens. . . . . From careful dissection, I am convinced that it is only a part of the fibres belonging to the *two* roots which are attached to the grey substance, and that a considerable number of threads are lost in the fibrous part of the chord. . . . . The exact mode of their connection, however, with this latter substance is not known." Only a part, therefore, of Mr. GRAINGER's statements is really correct; for though some of the fibres of the posterior roots may be confined in their course within the white columns, this is certainly not the case with regard to the anterior roots; and many of the fibres of the *posterior* roots, which appeared to him to be longitudinal, were probably those which take a very oblique course towards the grey substance. The fibres, also, of the third pair of cerebral nerves, of which some have been said by Mr. GRAINGER to be continuous with the white columns, may be all traced to the vesicular substance situated below the iter a tertio ad quartum ventriculum. Great merit, however, is due to Mr. GRAINGER for having made out so much with so little assistance.

The question, then, arises,—Do the fibres of the posterior roots which ascend longitudinally in the posterior fasciculi transmit impressions to the sensorium? In answer to this question it may be stated,—1st, that, even if they reach the brain, their number appears insufficient to convey such impressions from all parts of the body; and 2ndly, that the anatomical connection of the posterior columns exclusively with the cerebellum would, *à priori*, lead us to a negative conclusion, unless we regard this latter organ as the common centre of sensation, which we have no grounds for believing it to be. But the same anatomical connection of the posterior columns with the cerebellum clearly indicates that they establish *some* physiological relation between this organ and the spinal chord; and if the cerebellum were proved to be, according to the generally received opinion, a centre of *motor* power *only*, destined for the control and coordination of complex muscular movements, it would follow that these columns were motor, and not sensitive, whether or not we admit them to be constituted wholly, or in part, of the longitudinal fibres of the posterior roots*.

Upon anatomical grounds, then, it would appear, that the posterior white columns are not the channels of communication between the sensorium and the posterior spinal nerves. Experimental evidence tends to the same conclusion. Both VAN DEEN† and STILLING found that irritation of these columns excited no sensation

* That the cerebellum is in *some way* concerned in the regulation and coordination of muscular movements appears to follow from experiment and pathological investigation; but I think there are reasons for believing that the *mode of action* usually assigned to it is unsatisfactory, and at variance with many well-established facts. The discussion of this question, however, would be foreign to the present subject.

† Traités et Découvertes sur la Physiologie de la Moëlle Epinière. Leide, 1841.

when their connection with the grey substance was interrupted. Nothing, however, is more certain than that the posterior roots are in *immediate* connection exclusively with these columns, which are separated from the antero-lateral columns by the posterior lateral fissure, through which a few only of the roots proceed to the grey substance. It is difficult, therefore, to conceive how sensory impressions can be conveyed after destruction of the posterior columns, as *appears* to have taken place in cases put on record by several observers. The only explanation of which these cases seem to admit, is that either these columns were only partially destroyed, or that sensory fibres are contained also in the anterior roots.

The question, then, remains,—By what means *are* impressions which are received by the chord transmitted upwards to the brain?

By Volkmann it is believed that all, or nearly all the fibres of the nerve-roots terminate in the grey substance of the chord. In support of this opinion, he endeavours to show, both by measurement and weight, that there is no progressive increase, from below upwards, in the size of the white columns; that, on the contrary, he found these columns absolutely smaller in the cervical region than in the lower parts, and everywhere in direct proportion to the quantity of the grey substance. He further states, that the total area of all the spinal nerves cut transversely is many times greater than the area of a transverse section of the chord above the first cervical nerve. Opposed to that of Volkmann is the opinion of Kölliker*, who holds that the white columns of the chord are composed entirely of fibres derived from the spinal nerves. This opinion is founded on the fact pointed out by him,—and about the same time by myself,—that at least a large proportion of both roots, after passing through various parts of the grey substance, pursue a longitudinal course within the white columns. He further maintains, in opposition to Volkmann, that these columns do really increase progressively towards the brain, and that the nerve-fibres of which they mainly consist, do not recover their original diameter which they lost in passing through the grey substance. From these facts he calculates that the white columns are sufficiently large in the cervical region to give passage to all the spinal nerves, and that they form the channels of communication to and from the brain.

The results of my own observations are opposed to those of Volkmann, and partly in favour of those of Kölliker. In all the Mammalia examined by myself, I have found the quantity of the white substance of the chord absolutely greater in the cervical region than elsewhere. It cannot therefore be, as stated by Volkmann, in direct proportion to the grey substance, since the latter is far more abundant in the lumbar enlargement. Again, his opinion that all, or nearly all the spinal nerves terminate in the grey substance of the chord, is refuted by the fact, shown by both Kölliker and myself, that at least a great number of them do really pass out of the grey substance as longitudinal fibres of the white columns. But on the other hand,

* Mikroskopische Anatomie, Band ii.

I cannot agree with the statement of Kölliker, that all these fibres continue the same course within these columns in the direction of the brain; for it is certain, as I have already observed, that many of them not only pursue a *descending* course, but that others which actually *ascend* re-enter the grey substance to form loops. It is certain, also, that of the anterior roots in particular, the fibres, on reaching the grey substance, diverge in a direction as much *downwards* as upwards.

It seems therefore that many of the fibres which belong respectively to the anterior and posterior roots, in different regions of the chord, terminate there by forming with each other a series of loops, partly within the grey substance, and partly after extending through the latter into the white columns; and that these loops are of various sizes and lengths. Nor is it improbable that some of them may reach even as far as the brain, since it is well known that the formation of loops is at least one mode in which nerve-fibres terminate there. I am far from denying, however, that a portion of the roots may be connected with the vesicles of the chord.

Does the *grey substance* of the chord transmit impressions to and from the brain?—Stilling divided, in a living animal, the anterior white columns through their whole thickness, and found that voluntary power was still conveyed to parts below the section; but when the incision extended deeply into the grey substance, all voluntary movements were interrupted. From this experiment he infers that the grey substance alone is the conductor of voluntary power from the brain to the anterior roots; and the means of communication he explains by a system of fine longitudinal fibres which he found in the anterior grey substance. I thought formerly that I also had seen such fibres in comparatively small numbers; but further and more careful observation has led me to the conclusion, that excepting those in the *substantia gelatinosa* and those near the border of the *substantia spongiosa*, there is no regular system of fine longitudinal fibres in the grey substance. In looking for such fibres I have always selected a perfectly fresh chord. An exceedingly thin section, which by practice may be made with a very sharp instrument, was carefully laid on a glass slide and treated with acetic acid, which increases its transparency without producing any material alteration in the appearance of its fibres. It was then covered with thin glass, and examined, without any pressure, under sufficiently high powers. In such a section, fibres of various diameters—many exceedingly fine—may be seen to cross each other in every direction, but not in one direction more than another. It is probable that even the longitudinal fibres of the *substantia gelatinosa* are not continued, *individually*, as far as the brain, but pass out at intervals into the white columns, since their number, as well as the breadth of the *substantia gelatinosa*, does not increase regularly from below upwards. I believe that they are all derived from the fine bundles of the posterior roots*.

* Kölliker supposes that the nucleated fibres of the *substantia gelatinosa*, described by Remak, are processes of the nerve-vesicles; but these processes contain no nuclei. It is more probable that the fibres of Remak are really small blood-vessels, which have precisely the same appearance, and of which a large number enter with the posterior roots and pursue the same longitudinal course in the *substantia gelatinosa*.

The results obtained from the experiments of STILLING above described are involved in much obscurity; for in addition to other reasons, it would seem extremely difficult, if not impossible, to avoid leaving a portion of the anterior columns undivided,—that portion, namely, which is situated internal to the anterior grey substance. But even if they tended to show that the power of volition is not transmitted by these columns, they by no means prove it to be communicated by the grey substance, since a deep incision into the latter would divide not only many fibres of the anterior roots extending upwards, but a portion also of the lateral white columns into which these roots are prolonged.

That the grey substance of the chord does transmit impressions *from one side to the other, has been* fully proved by the experiments of VAN DEEN and STILLING; and the means by which the communication is effected is satisfactorily explained by the structure of the transverse commissure described by both STILLING and myself*. I believe, however, that the fibres of this commissure do not form a distinct set connecting the grey substance of one side with that of the other, but that they are all continuous with the roots of the nerves and the white columns†.

From certain facts, then, described in this communication, it would appear that the white columns are mainly constituted of fibres derived from both roots of the nerves. It is true that the processes of some of the vesicles situated near the margin of the grey substance may be seen to extend into these columns, with fibres of which they perhaps are continuous; and it is not improbable that others more deeply seated may have the same connection. But if there be, as some physiologists believe, a

* The structure immediately surrounding the spinal canal is described by STILLING as a *circular commissure* composed of fine grey fibres. By REMAK it is spoken of as the *commissura gelatinosa*, and as a continuation of the *substantia gelatinosa* of the posterior grey substance. KÖLLIKER, who declares that this structure was mistaken by STILLING for a spinal canal—the existence of which he denies—also considers it as a peculiar kind of vesicular substance, enclosing a central nucleus, (*grauer centraler Kern*) and which he calls the *substantia grisea centralis*. It appears, however, as described in my former communication, to consist of a circular layer of fine fibrous tissue surrounding and supporting the columnar epithelium which forms the wall of the spinal canal. The existence of this canal is unquestionable; but in the human chord it is often closed, and reduced, as if by lateral pressure, to a mere line; and around this the columnar epithelium is arranged in the form of an ellipse, which KÖLLIKER appears to have mistaken for a double nucleus.

† There seems to be a great correspondence in the fibrous arrangement between the grey substance of the spinal chord and the chiasma of the optic nerves. In a few transparent preparations that I have made of the latter, the structure first pointed out by MAYO is very clearly discerned; but the course of the fibres is found to be still more intricate. There is a remarkable circumstance that I have observed, and which I do not recollect to have seen recorded, respecting the cross action of the optic nerves. After looking through the microscope, if the eye employed be directed to other objects, it must be well known that the *definition* is found to be unimpaired, although the *colour* is rendered darker than natural, in direct proportion to the previous stimulus of light; but if the same objects be viewed by the other eye which has not been employed at the microscope, I have invariably noticed that an opposite effect is produced, and in the same proportion,—the *colour* remains *natural*, but the *definition* is more or less *impaired*. It may be stated then as a law, that, *By close observation with magnifying glasses, under the stimulus of a strong light,—in the eye employed, the perception of colour, but not the defining power, is affected; while in the opposite eye, the defining power, but not the perception of colour, is impaired, and in a direct and equal ratio.*

direct continuity between the cerebral fibres and the caudate vesicles of the chord, I am disposed to think it is established by means of the system of fine nerve-tubes which are seen to proceed from the grey substance of the latter as longitudinal fibres of the white columns. In opposition to WAGNER, I agree with KÖLLIKER in the opinion that the process of a nerve-vesicle never becomes the axis-cylinder of a double-contoured nerve-tube, and that what WAGNER considered as the process of a nerve-vesicle was probably the axis-cylinder itself. The evidence of any direct connection between these processes and the roots of the nerves is very unsatisfactory; for amongst thousands of preparations examined by myself with the greatest care, I met with a few cases only in which it was at all probable.

If any of those fibres of the posterior roots, which, after traversing the grey substance, enter the anterior white columns, could be proved to extend as far as the brain, the indications of sensation which are said to have followed irritation of these columns might be thus explained. It is quite certain, however, as already stated, that an equal number of those fibres pursue a *downward* course. On the other hand, if the anterior columns really be, as I believe they are, one of the channels which transmit downwards the influence of the will, they can perform this office only by first entering the grey substance, since into this all the fibres of the anterior roots may be traced. The same inference may be drawn with regard to the lateral white columns, which from their connection, through the grey substance, with the roots of the nerves, would appear to be both sensitive and motor.

The fact that many fibres of each root, on entering the grey substance, not only extend both upwards and downwards to a considerable distance beyond their point of entrance, but intermingle also in the most intricate manner with those of other roots, may serve to explain how impressions made at one particular spot are communicated in different directions to distant parts of the chord, so as to excite a simultaneous and sympathetic action in classes of muscles which otherwise would appear unconnected. It is probable that the fibres which quit the grey substance and return to it by the formation of loops within the white columns, may take that course for the purpose of stimulating particular parts of that substance, without affecting those which are intermediate.

In concluding this series of investigations, it has appeared to me, that, considering the beauty and transparency of my preparations, the distinctness with which their several parts are preserved, and the persevering labour which has been bestowed upon them, I might almost venture to think we have obtained nearly all that it is possible to know—with our present means—concerning the minute anatomy of the spinal chord. Our further progress is arrested chiefly by the continual and intricate deviations observed in the course of its fibres, and the consequent disorder in which its elements appear to be involved, when viewed in section. The confusion, however, is doubtless only apparent; for in this wonderful organ, with all its complexity of structure, we have no reason to doubt the constant and uniform existence of that

beautiful order and design so conspicuously manifested by organic life in the formation of every part which science has revealed to the inquiring eye. Every fibre, probably, has its own fixed and appointed course, in accordance with the particular function it is destined to perform.

Explanation of the Plates.

PLATE XXIII.

Represents a longitudinal section through the cervical enlargement of the spinal chord of the Cat, from the 8th to the 12th pair of nerves. Magnified 30 diameters*.

P, C. Posterior white columns. A, C. Anterior white columns. G. Grey substance between the white columns. P. Posterior roots of the nerves, consisting of three kinds,—*a*, *b* and *c*. The second kind, *b*, traverse the posterior columns at first obliquely towards the mesial line, that is, in a plane oblique to that of the section, and were therefore divided near the surface of the chord. A. Anterior roots of the nerves. A, C'. A portion of the anterior column, showing the arrangement of their longitudinal fibres.

PLATE XXIV.

Represents a longitudinal section through the lumbar enlargement of the spinal chord of the Ox. Magnified 30 diameters.

P, C. Posterior columns, showing the intricate interlacement formed by the posterior roots of the nerves. P. Posterior roots. A, C. Anterior white columns. A. Anterior roots of the nerves. G. The anterior and posterior grey substance traversed by the fibres of the anterior and posterior roots of the nerves. *g*. The *substantia gelatinosa*.

In both Plates the vesicles have been omitted in order to avoid obscuring the fibres, which were observed by means of Ross's half-inch object-glass, with No. 2 eye-piece, giving a power of 150 diameters.

* The outlines of the drawings were made by means of a power of 40 diameters, but were reduced nearly one-third by the engraver.

XIV. *On the Thermal Effects of Fluids in Motion.* *By* WILLIAM THOMSON, *M.A., F.R.S., F.R.S.E., &c., Professor of Natural Philosophy in the University of Glasgow, For. Mem. of the Royal Swedish Academy of Sciences; and* J. P. JOULE, *F.R.S., F.C.S., Corr. Mem. R.A. Turin, Vice-President of the Literary and Philosophical Society of Manchester, &c.*

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IN a paper communicated to the Royal Society, June 20, 1844, "On the Changes of Temperature produced by the Rarefaction and Condensation of Air*," Mr. JOULE pointed out the dynamical cause of the principal phenomena, and described the experiments upon which his conclusions were founded. Subsequently Professor THOMSON pointed out that the accordance discovered in that investigation between the work spent and the mechanical equivalent of the heat evolved in the compression of air may be only approximate, and in a paper communicated to the Royal Society of Edinburgh in April 1851, "On a Method of discovering experimentally the relation between the Mechanical Work spent, and the Heat produced by the compression of a Gaseous Fluid†," proposed the method of experimenting adopted in the present investigation, by means of which we have already arrived at partial results‡. This method consists in forcing the compressed elastic fluid through a mass of porous non-conducting material, and observing the consequent change of temperature in the elastic fluid. The porous plug was adopted instead of a single orifice, in order that the work done by the expanding fluid may be immediately spent in friction, without any appreciable portion of it being even temporarily employed to generate ordinary *vis viva*, or being devoted to produce sound. The non-conducting material was chosen to diminish as much as possible all loss of thermal effect by conduction, either from the air on one side to the air on the other side of the plug, or between the plug and the surrounding matter.

A principal object of the researches is to determine the value of μ, CARNOT'S function. If the gas fulfilled perfectly the laws of compression and expansion ordinarily assumed, we should have§

$$\frac{1}{\mu}=\frac{\frac{1}{E}+t}{J}+\frac{K\delta}{Ep_0u_0\log P},$$

where J is the mechanical equivalent of the thermal unit; p_0u_0 the product of the

* Philosophical Magazine, S. 3, vol. xxvi. p. 369.

† Transactions of the Royal Society, Edinburgh, vol. xx. Part II.

‡ Philosophical Magazine, S. 4, vol. iv. p. 481.

§ Dynamical Theory of Heat, equation (7), § 80, Transactions of the Royal Society of Edinburgh, vol. xx p. 297.

pressure in pounds on the square foot into the volume in cubic feet of a pound of the gas at 0° Cent.; P is the ratio of the pressure on the high pressure side to that on the other side of the plug; δ is the observed cooling effect; t the temperature Cent. of the bath, and K the thermal capacity of a pound of the gas under constant pressure equal to that on the low pressure side of the gas. To establish this equation it is only necessary to remark that $K\delta$ is the heat that would have to be added to each pound of the exit stream of air, to bring it to the temperature of the bath, and is the same (according to the general principle of mechanical energy) as would have to be added to it in passing through the plug, to make it leave the plug with its temperature unaltered. We have therefore $K\delta = -H$, in terms of the notation used in the passage referred to.

On the above hypothesis (that the gas fulfils the laws of compression and expansion ordinarily assumed) $\frac{\delta}{\log P}$ would be the same for all values of P; but Regnault has shown that the hypothesis is not rigorously true for atmospheric air, and our experiments show that $\frac{\delta}{\log P}$ increases with P. Hence, in reducing the experiments, a correction must be first applied to take into account the deviations, as far as they are known, of the fluid used, from the gaseous laws, and then the value of μ may be determined. The formula by which this is to be done is the following (Dynamical Theory of Heat, equation (f), § 74, or equation (17), § 95, and (8), § 89)—

$$\frac{1}{\mu} = \frac{\frac{1}{J}\{w-(p'u'-pu)\}+K\delta}{\frac{dw}{dt}},$$

where $w = \int_u^{u'} p\,dv$,

u and u' denoting the volumes of a pound of the gas at the high pressure and low pressure respectively, and at the same temperature (that of the bath), and v the volume of a pound of it at that temperature, when at any intermediate pressure p. An expression for w for any temperature may be derived from an empirical formula for the compressibility of air at that temperature, and between the limits of pressure in the experiment.

The apparatus, which we have been enabled to provide by the assistance of a grant from the Royal Society, consists mainly of a pump, by which air may be forced into a series of tubes acting at once as a receiver of the elastic fluid, and as a means of communicating to it any required temperature; nozles, and plugs of porous material being employed to discharge the air against the bulb of a thermometer.

The pump a, fig. 1, consists of a cast-iron cylinder of 6 inches internal diameter, in which a piston, fig. 2, fitted with spiral metallic packing (of antifriction metal), works by the direct action of the beam of a steam-engine through a stroke of 22 inches. The pump is single-acting, the air entering at the base of the cylinder during the up-stroke, and being expelled thence into the receiving tubes by the down-stroke. The governor of the steam-engine limits the number of complete

strokes of the pump to 27 per minute. The valves, fig. 3, consist of loose spheres of brass 0·6 of an inch in diameter, which fall by their own gravity over orifices 0·45 of an inch diameter. The cylinder and valves in connection with it are immersed in water to prevent the wear and tear which might arise from a variable or too elevated temperature.

Fig. 1.

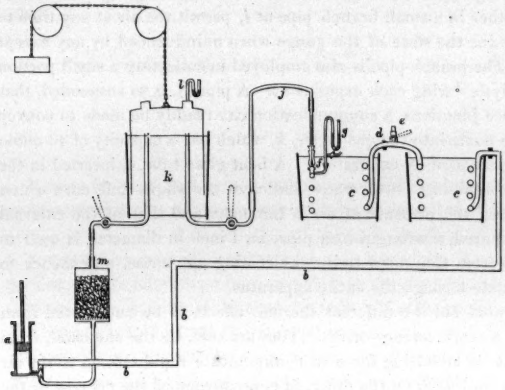


Fig. 2.

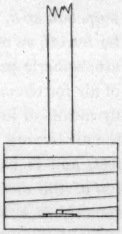


Fig. 3.

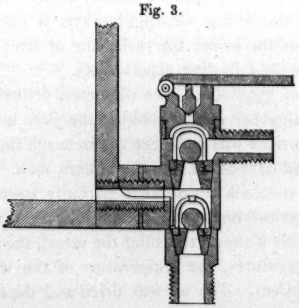
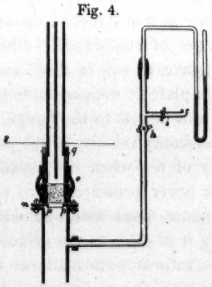


Fig. 4.



Wrought-iron tubing, *b b*, fig. 1, of 2 inches internal diameter, conducts the compressed air horizontally a distance of 6 feet, thence vertically to an elevation of 18 feet, where another length of 23 feet conveys it to the copper tubing, *c c*; the junction being effected by means of a coupling-joint. The copper tubing, which is of 2 inches internal diameter and 74 feet in length, is arranged in two coils, each being immersed in a wooden vessel of 4 feet diameter, from the bottom and sides of

which it is kept at a distance of 6 inches. The coils are connected by means of a coupling joint *d*, near which a stopcock, *e*, is placed, in order to let a portion of air escape when it is wanted to reduce the pressure. The terminal coil has a flange, *f*, to which any required nozle may be attached by means of screw-bolts. Near the flange, a small pipe, *g*, is screwed, at the termination of which a calibrated glass tube bent (as shown in fig. 4), and partly filled with mercury, is tightly secured. A stopcock at *h*, and another in a small branch pipe at *i*, permit the air at any time to be let off, so as to examine the state of the gauge when uninfluenced by any except atmospheric pressure. The branch pipe is also employed in collecting a small portion of air for chemical analysis during each experiment. A pipe, *j*, is so suspended, that by means of india-rubber junctions, a communication can readily be made to convey the air issuing from the nozle into the gas-meter, *k*, which has a capacity of 40 cubic feet, and is carefully graduated by calibration. A bent glass tube, *l*, inserted in the top of the meter, and containing a little water, indicates the slight difference which sometimes exists between the pressure of air in the meter and that of the external atmosphere. When required, a wrought-iron pipe, *m*, 1 inch in diameter, is used to convey the elastic fluid from the meter to the desiccating apparatus, and thence to the pump so as to circulate through the entire apparatus.

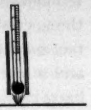
We have already pointed out the different thermal effects to be anticipated from the rushing of air from a single narrow orifice. They are *cold*, on the one hand, from the expenditure of heat in labouring force to communicate rapid motion to the air by means of expansion; and *heat*, on the other, in consequence of the *vis viva* of the rushing air being reconverted into heat. The two opposite effects nearly neutralize each other at 2 or 3 inches distance from the orifice, leaving however a slight preponderance of cooling effect; but close to the orifice the variations of temperature are excessive, as will be made manifest by the following experiments.

A thin plate of copper, having a hole of $\frac{1}{10}$th of an inch diameter, drilled in the centre, was bolted to the flange, an india-rubber washer making the joint air-tight. At the ordinary velocity of the pump the orifice was sufficient to discharge the whole quantity of air when its pressure arrived at 124 lbs. on the square inch. When however lower pressures were tried, the stopcock *e* was kept partially open. The thermometer used was one with a spherical bulb 0·15 of an inch in diameter. Holding it as close to the orifice as possible without touching the metal, the following observations were made at various pressures, the temperature of the water in which the coils were immersed being 22° Cent. The air was dried and deprived of carbonic acid by passing it, previous to entering the pump, through a vessel $4\frac{1}{2}$ feet long and 20 inches diameter, filled with quicklime.

Total pressure of the air in lbs. on the square inch.	Temperature Centigrade.	Depression below temperature of bath.
124	8°·58	13°·42
72	11·65	10·35
31	16·25	5·75

The heating effect was exhibited as follows:—The bulb of the thermometer was inserted into a piece of conical gutta percha pipe in such a manner that an extremely narrow passage was allowed between the interior surface of the pipe and the bulb. Thus armed, the thermometer was held, as represented by fig. 5, at half an inch distance from the orifice, when the following results were obtained:—

Fig. 5.



Total pressure of the air in lbs. on the square inch.	Temperature Centigrade.	Elevation above temperature of bath.
124	45°·75	23°·75
71	39·23	17·23
31	26·2	4·20

It must be remarked, that the above recorded thermal effects are not to be taken as representing the maximum results to be derived from the rushing air at the pressures named. The determination of these, in the form of experiment above given, is prevented by several circumstances. In particular it must be observed, that the cooling effects must have been reduced in consequence of the heat evolved by the friction of the rushing air against the bulb of the thermometer. The heating effects, resulting as they do from the absorption and conversion into heat of the *vis viva* of the rushing air, depend very much upon the narrowness of the space between the thermometer and gutta percha pipe. We intend further on to return to this subject, but in the mean time will mention three forms of experiment whereby the heating effect is very strikingly and instructively exhibited.

Experiment 1.—The finger and thumb are brought over the orifice, as represented in fig. 6, so that by gradually closing them the stream of air is pinched. It is found that the effort to close the finger and thumb is opposed by considerable force, which increases with the pressure applied. At the same time a strong tremulous motion is felt and a shrill noise is heard, whilst the heat produced in five or six seconds necessitates the termination of the experiment.

Fig. 6.

Experiment 2.—Fig. 7. The finger is placed over the orifice and pressed until a thin stratum of air escapes between the copper-plate and the finger. In this case the burning heat of the rushing air is equally remarkable in spite of the proximity of the finger to the cold metal.

Fig. 7.

Experiment 3.—Fig. 8. A piece of thick india-rubber is pressed by the finger over the narrow orifice so as to allow a thin stream of air to rush between the india-rubber and the plate of copper. In this case the india-rubber is speedily raised to a temperature which prevents its being handled comfortably.

Fig. 8.

We have now adduced enough to illustrate the immense and sudden changes of temperature which exist in the "rapids" of a current of air, changes which point out the necessity of employing a porous plug, in order that when the air arrives at the thermometer its state may be reduced to a uniform condition. Figs. 4 and 9 represent our first arrangement for the porous plug, where *n* is a brass casting with flange to bolt to the copper tube. It has eight studs, *o*, and eight holes, *pp*, drilled into the inner part of the flange. The studs and holes furnish the means of securing the porous material (in the present instance of cotton wool) in its place, by binding it down tightly with twine. Immediate contact between the cotton and metal is prevented by the insertion of a piece of india-rubber tubing; *qqq* are three pieces of india-rubber tube inserted within each other, the inner one communicating with a glass tube *r*, through which the divisions of the thermometer may be seen, and which serves to convey the air to the meter. In the experiments about to be given, the thermometer was in immediate contact with the cotton plug as represented in the figure, and the nozle was immersed in the bath up to the line *s*. The weight of the cotton wool in a dry state was 251 grs., its specific gravity 1·404, and being compressed into a space $1\frac{1}{3}$ inch in diameter and 1·9 inch long, the opening left for the passage of air must have been equal in volume to a pipe to 1·33 of an inch diameter.

Fig. 9.

First series of experiments. Atmospheric air dried and deprived of carbonic acid by quicklime. Gauge 73·6; barom. 30·04=14·695 lbs. pressure per square inch.

Gauge.	Total pressure in lbs. per square inch.	Cubic inches of air passed per minute reduced to atmospheric pressure.	Temperature of bath* ascertained by Thermometer No. 1, in Centigrade degrees.	Temperature of the issuing air, ascertained by Thermometer No. 2.	Cooling effect.
37·5 37·5 38 37·8 } 37·7	35·854	12703	445 445·5 445·9 446 } 445·6 = 18°·2676	414 414 414·6 414·8 } 414·35 = 17°·8298	0°·4378
38 38 37·8 38 } 37·9	35·647	12703	446·1 446·6 446·8 447·1 } 446·65 = 18·3128	415·4 416 416·8 417·6 } 416·45 = 17·9295	0·3833
38 37·75 37·5 37·5 } 37·69	35·866	12703	447·2 447·5 447·8 448 } 447·62 = 18·3545	418 418·2 418·4 418 } 418·15 = 18·0110	0·3435

A Liebig tube containing sulphuric acid, specific gravity 1·8, gained 0·03 of a grain by passing through it, during the experiment, 100 cubic inches of air.

* By varying the temperature of the water in which the coils were immersed, it was found that the temperature of the water surrounding the first coil exercised no perceptible influence, the temperature of the rushing air being entirely regulated by that of the terminal coil. However, the precaution was taken of keeping both coils at nearly the same temperature.

The observations above tabulated were made at intervals of two or three minutes. It will be observed that the cooling effect appeared to be greater at the commencement than at the termination of the series. This may be attributed in a great measure to the drying of the cotton, which was found to contain at least 5 per cent. of moisture after exposure to the atmosphere. There was also another source of interference with the accuracy of the results owing to a considerable oscillation of pressure arising from the action of the pump. We had remarked that when the number of strokes of the engine was suddenly reduced from twenty-seven to twenty-five per minute, a depression of the thermometer equal to some hundredths of a degree Cent. took place, a circumstance evidently owing to the entire mass of air in the coils and cotton plugs suffering dilatation without allowing time for the escape of the consequent thermal effect. Hence it was found absolutely essential to keep the pump working at a perfectly uniform rate. For a similar reason it was also most important to prevent the oscillations of pressure due to the action of the pump, particularly as it appeared obvious that the heat evolved by the sudden increase of pressure, on the admission of a fresh supply of air from the pump, would arrive at the thermometer in a larger proportion than the cold produced by the subsequent gradual dilatation. In fact, on making an experiment in which the air was kept at a low pressure, by opening a stopcock provided for the purpose, the oscillations of pressure amounting to $\frac{1}{10}$th of the whole, it was found that an apparent heating effect, equal to 0°·2 Cent., was produced instead of a small cooling effect.

It became therefore necessary to obviate the above source of error, and the method first employed with that view, was to place a diaphragm of copper with a hole in its centre $\frac{1}{7}$th of an inch in diameter at the junction between the iron and copper pipes. The oscillation being thus reduced, so as to be hardly perceptible, we made the following observations.

Second series of experiments. Atmospheric air dried and deprived of carbonic acid by quicklime. Gauge 73·75; barometer 30·162=14·755 lbs. pressure per square inch; thermometer 19°·3 Cent.

Gauge.	Total pressure in lbs. per square inch.	Cubic inches of air issuing per minute at atmospheric pressure.	Temperature of bath by Thermometer No. 1, degrees Centigrade.	Temperature of issuing air by Thermometer No. 2, degrees Centigrade.	Cooling effect.
39 38·6 38·5 38·5 } 38·65	36·069	11796	467 467 467 467·1 } 467·02=19°·186	434·6 435 435 435 } 434·9 =18°·810	0°·377
38·5 38·8 38·8 38·75 } 38·79	35·912	11796	467·1 467·2 467·2 467·3 } 467·2 =19·194	435·1 435·4 435·6 435·4 } 435·37=18·832	0·362
38·8 38·8 38·8 38·8 } 38·8	35·900	11796	467·3 467·4 467·4 467·4 } 467·37=19·202	435·6 435·8 435·9 436 } 435·82=18·854	0·348

Suspecting that particles of the sperm oil employed for lubricating the pump were carried mechanically to the cotton plug and interfered with the results, we now substituted a box with perforated caps, filled with cotton wool, for the diaphragm used in the last series. With this arrangement the pressure was kept as uniform as with the other, and all solid and liquid particles were kept back by filtration.

Third series of experiments. Atmospheric air dried and deprived of carbonic acid by quicklime*, and filtered through cotton. Gauge 73·7; thermometer 21°·7 Cent.; barometer 30·10=14·71 lbs. on the square inch.

Time of observation.	Gauge.		Total pressure in lbs. per square inch.	Cubic inches of air issuing per minute at atmospheric pressure.	Temperature of bath by Thermometer No. 1, in degrees Centigrade.		Temperature of the issuing air by Thermometer No. 2, in degrees Centigrade.		Cooling effect.
m									
3	39				357·7		337·35		
6	39·1	39·2	34·410	11784	357·8	357·92=14·506	337·8	337·89=14·183	0·323
9	39·5				358		338		
12	39·2				358·2		338·4		
15	39·1				358·7		338·8		
16	39·35	39·19	34·418	11784	358·9	358·97=14·552	338·7	338·85=14·230	0·322
18	39·1				359·1		339		
21	39·2				359·2		338·9		
23	39·2				359·4		339·25		
25	39·1	39·18	34·426	11784	359·7	359·72=14·584	339·8	339·69=14·270	0·314
28	39·2				359·8		339·7		
30	39·2				360		340		
32	39·5				360·1		340		
34	39·3	39·34	34·279	11784	360·2	360·27=14·607	340·2	340·25=14·296	0·311
36	39·25				360·4		340·4		
38	39·3				360·4		340·4		

* The use of quicklime as a desiccating agent was suggested to us by Mr. Thomas Ransome. It answered its purpose admirably after it had fallen a little by use, so as to be finely subdivided. The perfection of its action was shown by the desiccating cylinder remaining, after having been used two hours, cold at the lower part, while the upper part for about 9 inches was made very hot. The analysis of the air passed during the third series of experiments showed that one of the Liebig tubes had gained no weight whatever; and in one instance we have observed that the sulphuric acid of 1·8 specific gravity, actually lost weight, apparently indicating that the air dried by quicklime was able to remove water from acid of that density.

The stopcock for reducing pressure being now partially opened, the observations were continued as follows:—

Time of observation.	Gauge.	Total pressure in lbs. per square inch.	Temperature of bath by Thermometer No. 1, in degrees Centigrade.	Temperature of the issuing air by Thermometer No. 2, in degrees Centigrade.	Cooling effect.
h m					
50	55·1		361·7	344	
52	55·1		361·9	344·8	
54	55·1		361·9	345·3	
55	55·1		361·9	345·8	
57	55·1		362·1	346·0	
59	55·1	55·12 22·876	362·3 362·26 = 14°693	346·4 346·19 = 14°579	0°114
1 1	55·1		362·4	346·9	
3	55·1		362·7	347·2	
5	55·1		362·7	347·6	
7	55·3		363	347·9	
11	54·3		363·3	348·9	
13	54·4		363·3	348·9	
15	54·4		363·5	349·2	
17	54·7		363·7	349·4	
19	54·5	54·51 23·217	363·9 363·82 = 14·760	350 349·74 = 14·749	0·011
20	54·5		364·1	350	
22	54·6		364·2	350·3	
24	54·6		364·2	350·4	
26	54·6		364·2	350·6	
30	54·6		375	356·4	
32	54·6		375·4	358·2	
33	54·2		375·4	359·4	
35	54·3		375·5	359·8	
37	54·4		375·8	360	
39	54·6	54·38 23·277	375·7 375·7 = 15·270	360·1 360 = 15·238	0·032
40	54·3		375·8	360·3	
42	54·5		376	360·4	

During the above experiment 100 cubic inches of the air was slowly passed through two Liebig tubes containing sulphuric acid, specific gravity 1·8. The first tube gained 0·006 of a grain, the second remained at exactly the same weight.

P.S. Oct. 14, 1853.—The apparently anomalous results contained in the last Table have been fully explained, and shown to depend on the alteration of pressure which took place towards the beginning of the interval of time from 42^m to 50^m, by subsequent researches which we hope soon to lay before the Royal Society.

XV. *On the Anatomy and Physiology of* Cordylophora*, *a contribution to our knowledge of the Tubularian Zoophytes.* By George James Allman, *M.D., M.R.I.A., Professor of Botany in the University of Dublin, and Examiner in Zoology and Botany in the Queen's University in Ireland. Communicated by Professor* Edward Forbes, *F.R.S.*

Received May 31,—Read June 16, 1853.

THOUGH the attention of several physiologists has been directed to the Tubularian Zoophytes, and though the elucidation of many points of interest in their structure and in the physiological phenomena presented by them has been the result, yet in numerous most important particulars our knowledge of the *Tubulariadæ* is still very imperfect, notwithstanding the great value which a more thorough acquaintance with these simple organisms must possess in its bearing upon some of the leading questions in physiology.

It was with the view of filling up some of the deficiencies which still exist in our knowledge of the lower zoophytes, that the following researches were undertaken; and *Cordylophora lacustris*, affording as it does a fine typical example of tubularian structure, and being easily kept alive for months, was deemed peculiarly favourable for prolonged and careful observation; the results, moreover, which have been obtained from the investigation of this zoophyte, have been in many instances confirmed and extended by corresponding inquiries, instituted at the same time into the structure of allied genera.

It is certain that considerable confusion as well as much cumbrous circumlocution might be avoided, by giving greater precision and completeness to the terminology employed in the description of structure among zoophytes. With this view, I propose in the first place to employ the word *polype* in its most restricted sense.

* The genus *Cordylophora* may be characterized as follows:—

Fam. Tubulariadæ.

Genus Cordylophora, Allman.

Char. Gen.—*Polypi* tentaculis numerosis sparsis teretibus. *Capsulæ genitales* subsessiles in ramulis ultimis pone polypos affixæ. *Polyparium* pergamentaceum, ramosum, stolone fistuloso repente fixum. *Embryo* liber, subcylindricus, universè ciliatus.

Nomen.—Κορδύλη clava et φορέω fero.

Species unica, *C. lacustris*.

Habitat.—In aquis dulcibus quietis, corpora varia submersa obducens, et locos obscuros amans.

This word is used ambiguously by writers, being often employed to designate the entire composite fabric, the aggregate result of gemmation; while at other times it is intended to indicate each of those peculiar organisms which, almost always furnished with a mouth and tentacula, are developed upon various points of a common living basis, and are eminently characteristic of zoophytic form. In the following paper I shall use the term *polype* strictly in the latter signification; and for the entire mass, whether consisting of a single polype, as in *Hydra*, or of many united into a more or less definite assemblage, it will be sufficiently convenient to employ merely the term *zoophyte**.

Besides the more definite limits within which it is thus necessary to confine the term polype, the requirements of precise description demand the use of a few additional terms. To the common living basis by which the several polypes in a composite zoophyte are connected with one another, I propose to give the name of *cœnosarc*†, and every composite zoophyte will thus consist of a variable number of polypes, developing themselves from certain more or less definite points of a common cœnosarc. The term *polypary* has been used with just as little precision as *polype*, being sometimes employed to express this common connecting basis, and at other times being applied exclusively to the solid protective structures, whether forming for the zoophyte an external covering or constituting an internal axis; the ambiguity which thus results will be got rid of by using the word cœnosarc as here defined, and restricting the term polypary to the solid protective structures of the zoophyte. All the hydroid zoophytes can be proved to consist essentially of two distinct membranes; to the external of these membranes I shall give the name of *ectoderm*‡, and to the internal that of *endoderm*§.

ORGANS FOR THE PRESERVATION OF THE INDIVIDUAL.

1. *Ectoderm and Polypary.*

(*a.*) *Ectoderm.*—The ectoderm (Plates XXV. and XXVI. figs. 3, 4, 9*a*, *a*, *a*) is a well-defined membrane; it is composed of cells, and forms the external layer of the polypes and cœnosarc. Multitudes of *thread-cells* are developed in the substance of the ectoderm. The thread-cells, when in a quiescent state (fig. 5), present the appearance of minute ovate capsules, slightly curved at one end, and with a transparent cylinder occupying about two-thirds of the axis. Under the influence of excitement, one extremity of

* In the following paper the term *zoophyte* is thus used in its restricted application, by which it is confined to the true polype-bearing *Radiata*, and no more convenient or expressive word can be employed for the purpose. If we except the changes which modern research has rendered necessary in removing from the zoophytes of the older authors, the Sponges, Corallines and *Polyzoa*, this term was used long ago in almost exactly the same sense by Pallas, and by Ellis and Solander, and some of our best living zoologists are now employing it with similar limitations.

† κοινὸς, σάρξ. ‡ ἐκτὸς, δέρμα. § ἔνδον, δέρμα.

the capsule becomes suddenly prolonged into a conical projection, surrounded near its apex by a circle of very minute curved spicula, the capsule at the same instant appearing empty (fig. 6). This phenomenon seems to consist in the sudden eversion through one end of the capsule of a delicate sac which had previously lain invaginated within it, and is generally immediately followed by the projection of a long and fine filament from the free end of the everted sac. The structure of these thread-cells, and the phenomena consequent on excitement, closely resemble what we find in the "hastigerous organs" of *Hydra*. The greater minuteness however of the thread-cells in *Cordylophora* renders it much more difficult to obtain a satisfactory examination of their structure. The thread-cells are developed in the interior of mother-cells (fig. 7), which are themselves secondary cells formed within the ectodermal cells; but I was not able to determine how far their development devolves exclusively upon certain cells of the ectoderm specially devoted to this office. The ectoderm of the tentacula differs slightly from that of the rest of the zoophyte, in the fact of the thread-cells being for the most part collected into groups, which present the appearance of little wart-like excrescences resembling the disposition of the corresponding organs in the tentacula of *Hydra*. At the extremity of the tentacula numerous thread-cells are crowded together, but there is no approach to the capitate terminations which are so striking in *Syncoryne*, and which, among other characters, zoologically distinguish that genus from *Cordylophora*.

(*b*.) *Polypary*.—The polypary (fig. 3 *b*, *b*) is a cylindrical tube investing the stems and branches of the zoophyte, and terminating just behind a short fleshy neck which immediately supports the polypes. On the younger branches it forms a thin, and almost colourless pergamentaceous investment; but in older parts of the zoophyte it is of a yellowish-brown colour, and composed of numerous layers. At the origin of the branches from the main stem, the polypary presents a number of annular corrugations. The cavity of the polypary appears by no means accurately filled by the cœnosarc, and we almost always find a considerable interval between the ectoderm of the cœnosarc and the walls of the polypary; this interval is crossed transversely by numerous processes of the ectoderm (fig. 3 *c*, *c*). It seems, however, almost certain that a very delicate living membrane is in actual contact throughout with the inner surface of the polypary, at least in all its younger portions; and the space which appears to exist between the polypary and the cœnosarc is then nothing but a large *lacuna* of the ectoderm, crossed by fleshy processes, which keep up a communication between its opposite walls. I have distinctly demonstrated the presence of such a membrane in *Syncoryne*, where I have traced the formation of the space in question from a simple lacuna of the ectoderm, though I have not succeeded in so decidedly making out these points in *Cordylophora*. No trace of organization can be detected in the polypary, which in all respects resembles a mere secretion deposited in layers from the ectoderm. Where the polypary terminates anteriorly, an exceedingly delicate transparent pellicle may be traced in continuity with it over

the neck and clavate body of the polype, at least as far as the roots of the posterior tentacula.

It is certain that the cavity of the polypary increases in diameter, up to a certain point, with the growth of the animal, a fact which we can reconcile with the account just given of its formation, only by attributing to it a capacity of distension subsequently to its original deposition.

2. *The Endoderm and its Cavity.*

The endoderm (fig. 3, *d*, *d*) constitutes a very distinctly cellular layer; it is in contact throughout with the ectoderm, but is nevertheless distinguishable from it by a very decided boundary, and can only be considered as directly continuous with it at the mouths of the polypes. It forms the walls of an uninterrupted cavity, which, extending through the axis of the entire zoophyte, opens externally at the polype-mouths. This cavity may be divided into four distinct regions, namely, the *post-buccal cavity*, the *stomach*, the *tentacular canals*, and the *canal of the cœnosarc*, a division which, though in some degree arbitrary, will be found very useful in description.

(*a*.) *Post-buccal cavity*.—The post-buccal cavity (fig. 3 *f*) is situated immediately behind the mouth (fig. 3 *e*), which is a simple unarmed orifice placed at the extremity of a conical projection, into which the body of the polype is continued anteriorly. The post-buccal cavity lies in the axis of this projection; its walls are formed of elongated cells, with their long diameter perpendicular to the surface, and its interior is quite destitute of rugæ. It is usually in the form of a narrow tube, and is capable of complete obliteration by the temporary approximation of its walls; it is sometimes, however, dilated into a globular space, and sometimes spread out by a kind of semi-eversion into a nearly flat disc, which I have seen employed by the animal as an organ of adhesion.

(*b*.) *Stomach*.—The stomach (fig. 3 *g*) occupies the whole of the interior of the clavate body of the polype; as the endoderm passes backward from the post-buccal cavity to constitute the walls of the stomach, we find it forming a peculiar tissue composed of elongated cells, whose rounded ends project in prominent masses into the cavity of the stomach, where they constitute large irregular rugæ. The tissue of these rugæ is very remarkable; the cells composing it (fig. 4 *b*) are mother-cells, giving origin in their interior to several free secondary cells with distinct nuclei. In some of these secondary cells the contents are colourless and transparent, and the nuclei are then very evident (fig. 4 *c*); others contain a brown granular matter, by which the nucleus is more or less obscured (fig. 4 *d*); while others may occasionally be found filled with a brood of young cells (fig. 8). Lying apparently free in the mother-cells may also be generally seen, besides the free secondary cells, some rather irregular masses of a deep brown granular substance (fig. 4 *e*). I have found in *Hydra* a structure in all its essential characters entirely similar to this, and we cannot refuse to recognise in it an example of true glandular structure. The secondary cells are

obviously true secreting cells, destined by their own rupture and that of the mother-cells to discharge their contents into the cavity of the stomach; while the free granular masses in the interior of the mother-cells were most probably originally contained in secondary cells, of whose secreting action they are the product, and which at a subsequent period had disappeared.

(*c.*) *Tentacular canals.*—The tentacula are hollow processes from the sides of the polypes, and their cavities are simple continuations of that of the stomach. This view is certainly opposed to the appearances presented on a superficial examination, and is at variance with the accounts given by other observers of the structure of the tentacula in the marine *Hydroida*; very careful and repeated observations, however, have convinced me of its truth. It must be admitted, that at first sight, the tentacula, even under well-managed microscopical examination, have exactly the appearance of tubes whose cavity is interrupted at regular intervals by completely formed transverse septa; I have, however, satisfied myself that in *Cordylophora*, and probably also in all the other *Hydroida*, the tube of the tentacula is perfectly continuous. The tentacula consist in reality, like all other parts of the animal, of an external or ectodermic layer (fig. 9 *a*, *a*), which is a simple continuation of the general ectoderm of the body, and of an internal or endodermic layer (fig. 9 *b*, *b*), which is in the same way a continuation of the endoderm of the stomach. The tentacular ectoderm has been already described. The endoderm consists of rather large cells with very delicate walls; it constitutes a thick lining of the tube, and nearly fills up the entire cavity; and indeed this cavity, by a temporary approximation of its walls, appears capable of occasional obliteration. The cells of the tentacular endoderm are glandular, and may be generally seen to contain the peculiar brown granules characteristic of the general endodermic secretion. The appearance of transverse septa is probably due to the occurrence at regular intervals, of interruptions in the continuity of the endodermic layer.

Though the tentacula of *Hydra* present no appearance of the transverse septa, their structure is nevertheless essentially the same as that of the tentacula of *Cordylophora*, the only difference being in the fact that the endoderm forms a thinner stratum in *Hydra*, and thus encroaches less on the cavity of the tube, and does not present the successive interruptions which in *Cordylophora* give rise to the fallacious appearance of transverse septa.

(*d.*) *Canal of the Cœnosarc.*—The cavity of the stomach is continuous posteriorly with the canal of the cœnosarc, fig. 3 *h*. The rounded masses of cells constituting the large rugæ into which the endoderm of the stomach is thrown, cannot be traced further back than the posterior extremity of the clavate body of the polype, and from this point, which may be conveniently though somewhat arbitrarily assumed as the posterior termination of the stomach, the canal of the cœnosarc commences, and thence extends as a continuous tube through all the ramifications of the cœnosarc. In that portion of the canal which immediately succeeds to the stomach, and which

is here much contracted, the endoderm resembles in all respects that of the stomach, except in the absence of distinct rugæ; but as the cœnosarc passes backwards under cover of the polypary, the cells lose their elongated form, the endoderm becomes thinner and the canal proportionately wider. Throughout all the rest of the cœnosarc the endoderm is formed of a compact tissue destitute of rugæ, but whose component cells continue to retain their glandular character (fig. 10).

The canal of the cœnosarc is filled with a fluid containing globular and irregular corpuscles of various sizes; these corpuscles may be observed to exhibit a kind of circulatory movement, which however is very irregular; they may be sometimes seen running from below upwards and again turning round in order to take a retrograde course; the motion of each corpuscle is generally confined to a small circuit, frequently not exceeding the diameter of the canal; sometimes many of the corpuscles stop altogether and allow others to pass them, while sometimes no motion whatever can be observed in any of the contents. The corpuscles while moving in their linear paths have besides a peculiar vibratile or tremulous motion in themselves, and this motion may generally be witnessed while the corpuscles are otherwise entirely at rest. The motion of the corpuscles may be traced all through the stems and branches, and even into the stomach. The cause of this phenomenon is very obscure; the existence of vibratile cilia on the walls of the canal might be adduced as an explanation, but these organs have never been made the subject of direct observation, notwithstanding the most careful attempts to demonstrate them; it appears to me that the true cause is to be found in currents produced by the active vital processes going on in the secreting cells of the endoderm, processes which we can hardly imagine to take place without causing local changes in the chemical constitution of the fluids in immediate contact with the cells, and a consequent disturbance of the stability of these fluids.

3. *Muscular System.*

Under this head may be described the only specialized motor apparatus which these animals present. It consists of numerous longitudinal fibres, which are in close contact with the inner surface of the ectoderm. These fibres (fig. 3 *i*, *i*, fig. 4 *f*, fig. 9 *c*) may be seen in all the naked portion of the cœnosarc, and in the body of the polype, and may be thence easily traced into the tentacula, in which they can be followed to the remote extremities of these organs, but they disappear where the cœnosarc passes under cover of the polypary. There can be little doubt that we are correct in considering them as the representatives of a muscular system, and in attributing to them the chief share in effecting the various changes of form which the polypes are perpetually presenting. Similar fibres may be witnessed in *Coryne*, *Syncoryne*, and other marine *Tubulariadæ*, in some of which they are even more distinct than in *Cordylophora*. In none could I detect the least trace of transverse striæ.

ORGANS FOR THE PRESERVATION OF THE SPECIES—EMBRYOLOGY.

The researches of modern naturalists, especially of Ehrenberg, Loven, Van Beneden, Sars, and Dujardin, have thrown much light on the embryology of the inferior zoophytes, and their observations have brought to our knowledge many remarkable and unexpected facts, though physiologists are by no means of one mind as to the true signification of several of the facts with which we have thus become acquainted.

During the later summer and autumn months *Cordylophora lacustris* may be seen with numerous oval capsules borne upon the ultimate or polypiferous ramuli, fig. 2. Each ramulus generally carries one, two, or three of these capsules, which are situated alternately on either side of the ramulus at some distance behind the body of the polype, and where the branch is still covered by the polypary, the more advanced capsules being always nearer to the main stems. They are nearly sessile on the supporting branches, and when fully developed are somewhat larger than the club-shaped bodies of the polypes. They are surrounded externally by a delicate continuation of the polypary, and as they approach towards maturity are found to contain within them either ova or spermatozoa.

The structure and development of these capsules and their contents are exceedingly interesting. The capsule on its first appearance presents itself as a small tubercle, consisting of a simple hernia-like protrusion, or diverticulum of the cœnosarc (fig. 11 *a*), enveloped by a delicate production of the polypary (fig. 11 *b*), and as the cavity of the diverticulum continues to communicate with the common canal of the branch, the contents of the latter pass freely into its interior. As the tubercle increases in size, we perceive that the external investment from the polypary immediately surrounds a sac of distinctly cellular structure, having the diverticulum from the cœnosarc projecting into it below, and having a remarkable system of branched tubes developed on its internal surface (figs. 12, 13, 14). These tubes spring from the sides of the diverticulum with whose cavity their own communicates; they occasionally inosculate with one another, and passing upwards in close connection with the walls of the cellular sac, extend to the summit of the sac where their branches terminate in cæca; they are lined with a layer of cells, which contain secondary cells and brown granules, and which thus resemble the general endoderm of the zoophyte. Thread-cells occur in the walls of the cellular sac, which is plainly referable to the ectodermal system. Between the cellular sac and the diverticulum is a considerable interval, which is filled with a granular fluid.

We next find that in those capsules which are to contain ova a number of spherical dark grey or bluish bodies (fig. 14 *f*) have become apparent in the midst of the granular fluid: these are *ova*; in most of them we may perceive the germinal vesicle in the form of a minute clear point, but the germinal spot is not apparent. The germinal vesicle soon disappears, and when the ova are now examined, the pheno-

menon of yelk-cleavage may be distinctly perceived in them, fig. 16. In the mean time a delicate structureless pellicle (fig. 13 *e*) has been formed on the external surface of the cellular sac; it is evidently a mere secretion from the surface of the sac, resembling the secretion of the polypary from the cœnosarc. Soon after the commencement of segmentation of the vitellus, the cellular sac with its system of ramified tubes disappears, and the cluster of ova is now distinctly exposed (fig. 15) lying upon the diverticulum from the cœnosarc, and immediately enclosed by the delicate structureless sac formed by the pellicle which had been secreted on the outside of the cellular sac, while the whole is included within the external pergamentaceous investment from the polypary. The ova vary in number; in many capsules I have counted ten, twelve, or even more, while some few contained only two or three.

The mulberry-like condition of the ova at length disappears, and these bodies now begin to elongate themselves, and when viewed by transmitted light, present a transparent margin (fig. 17). As development advances the ova become more and more elongated, and they soon exhibit a slow but evident motion in the interior of the containing sac (fig. 18). They are now in the condition of free embryos, only waiting for the rupture of the enclosing structures to escape into the surrounding medium and enjoy an independent existence. We accordingly soon find that these structures no longer confine them, and the embryos may now be seen escaping from the torn summit of the capsule in the form of infusorial animalcules clothed with short cilia, and swimming freely through the surrounding water (fig. 19).

The embryos on their escape from the capsule are usually of an elongated oval figure (figs. 19, 20), but very contractile, and capable of undergoing considerable change of form; they frequently assume a pyriform shape (fig. 21), but they never appear compressed, and the comparison with a *Planaria*, to which Dalyell and others have likened the embryos of *Campanularia*, will not here strictly apply. They are included in a distinct ectoderm, and contain in their interior a cavity which is separated from the ectoderm by a cellular interval representing the endoderm of the adult zoophyte, but in which the brown granules have not yet shown themselves. No mouth is yet evident, and the ectoderm seems in this early stage destitute of thread-cells.

After the embryo has continued for some time in the condition now described, the period of its final fixation approaches; the cilia disappear; one extremity becomes expanded into a kind of disc, by which it soon attaches itself to some fixed object; the mouth has now become apparent, and thread-cells are developed in the ectoderm (fig. 22); the embryo increases in length and thickness, and from the free extremity, which has begun to assume a clavate form, a single series of tentacula soon shoot forth (fig. 23); these are about four in number, and are situated a little behind the anterior end; the portion of the body which lies before them becomes the post-buccal cavity of the adult; the gastric cavity has become prolonged through the young stem; a delicate polypary has begun to invest it; and its walls begin to be

coloured by the characteristic secretion of the endoderm; other tentacula soon shoot forth behind those first formed, and the little *Cordylophora* resembles in all respects, except in size, a full-grown stem with a solitary polype. Its solitary condition however is not long retained; prolongations have already been sent out from its base; these attach themselves to the body on which it grows, and constitute the system of prostrate adherent tubes, while *gemmæ* are soon formed from these tubes and from the free stem, and convert the young single-polyped zoophyte into the adult many-polyped *Cordylophora**.

But besides the ovigerous capsules, a second kind (fig. 24) is developed on certain branches. These differ neither in situation nor in visible organization from the capsules destined to contain ova; their component sacs, internal diverticulum, and ramified canals entirely correspond with those of the ovigerous capsules; but instead of containing ova, they are filled with a turbid fluid, in the midst of which innumerable minute corpuscles may be seen to exhibit a peculiar vibratory movement occasionally visible under slight pressure through the transparent walls, and the same motion is continued in these corpuscles after they are liberated from the containing capsule and spread over the field of the microscope.

That these capsules represent a male system with spermatozoa, there cannot be the least doubt. When liberated from the capsule, the moving bodies show themselves under two different forms; they are either oval corpuscles with an excessively delicate caudal filament (fig. 24 *c*), or they are minute spherical cells (fig. 24 *b*) with a similar caudal filament (by whose undulations the cells are moved about through the surrounding fluid), and enclosing a nucleus-like corpuscle, with which the filament appears to be connected. The former would seem to be the spermatozoa in their free and fully developed condition, while the latter appear to be "vesicles of evolution," in which the body of the spermatozoon is still confined, the tail alone being disengaged. It still remains difficult to explain the mode in which the spermatozoa gain access to the ova. In some cases I have succeeded, under slight pressure, in forcing out the contents through a minute orifice which made its appearance in the summit of the capsule, but this orifice, notwithstanding its definite position, was probably the result of rupture, and I could not detect anything corresponding to it in the ovigerous capsules. It would perhaps be more in accordance with analogy to suppose that the spermatozoa make their way into the cavity of the diverticulum, that they are thus conveyed into that of the cœnosarc, and ultimately reach the ova through the diverticulum of the ovigerous capsules. The male and female capsules appear to be always borne on different stems.

* See an interesting paper by the Rev. Thomas Hincks, entitled 'Further Notes on British Zoophytes, with Descriptions of New Species,' published in the Annals and Magazine of Natural History, March 1853.

GENERAL CONSIDERATIONS.

From the description now given, it is manifest that *Cordylophora* (and there is little doubt that we are justified in extending the generalization to all the hydroid zoophytes) consists essentially of two distinct membranes, enclosing a cavity which opens externally by one or more orifices (polype-mouths), and which is prolonged into the interior of a variable number of filiform processes (tentacula)*. The two membranes appear to be essentially secreting structures, but the products of their respective secretory action strikingly differ from one another; for while the cells of the endoderm give origin to secondary cells which secrete a coloured granular substance, which may be fairly assumed as representing the biliary secretion of higher animals, the ectodermal cells give origin to cells of a very remarkable kind, whose characteristic secretion is a peculiar filament, whose import and office are as yet but very imperfectly understood.

In none of these zoophytes can any approach to a specialized circulatory system be discovered; the function of circulation appearing to be represented merely by a gradual transmission of the fluids from cell to cell, depending wholly on the permeability of tissue. A respiratory system is equally without any special representative. Whatever aërating influence the surrounding water exerts must be received through the entire surface of the zoophyte, though doubtless most powerfully through the portion not covered by the polypary. The muscular system has begun to be specialized, but no trace of a nervous system can yet be detected.

The existence of an extensive system of *lacunæ*, as described by Dujardin†, in some marine *Tubulariadæ*, and by Ecker‡ in *Hydra*, where this anatomist describes them as constituting a network of ramified canals, has not been confirmed by my own researches.

The true signification of the reproductive capsules of the hydroid zoophytes is a matter of considerable interest. On this subject much additional light will be thrown by comparing the embryological phenomena of *Cordylophora* with those which are known to occur in the allied genera of the sea. In the marine *Tubulariadæ*, two distinct kinds of bodies are developed besides the ordinary terminal polypes. Of these one consists in medusa-like bodies, which are generally developed at the bases of the tentacula§, and which after a time have been seen to detach themselves from the supporting polype and swim freely away. The other kind of bodies met with in these polypes are peculiar sacs, occupying a similar position to that of the medusoid

* The composition of the hydroid zoophytes out of two distinct membranes has been already pointed out by Huxley (Philosophical Transactions, 1849, Part II.), who has shown that the organs of the *Medusæ* are in the same way essentially composed of two distinct membranes. A very important affinity of structure is thus established between the hydroid zoophytes and the *Medusæ*.

† Ann. Sci. Nat. 3me sér. tome iv. 1845.

‡ Ann. des Sc. Nat. 3me sér. tome 10, 1848.

§ In *Eudendrium* the medusoid bodies are situated at a short distance behind the terminal polypes, and in *Perigonymus*, Sars, they are borne upon every part of the stems and branches.

bodies, but never detaching themselves from the parent, and always after a time filled with ova or spermatozoa*.

In *Campanularia* bodies of highly developed medusan conformation are produced, generally in considerable numbers, in external capsules situated on parts of the polypary always at a distance from the terminal polypes. When arrived at a certain degree of maturity, they escape from the capsule and swim freely through the surrounding water. In the capsules of *Campanularia* another kind of body has been also observed, in which the medusan structure is less complete, and these, after disengaging themselves from the cavity of the capsule, continue attached to its mouth, never becoming entirely free; while a third kind of bodies, corresponding with the fixed sacs of the *Tubulariadæ*, is also borne in the capsules of *Campanularia*†.

In some instances the free medusoid bodies of the *Tubulariadæ* have been found to contain ova, as has been observed by Wagner‡ in a tubularian polype from the Adriatic, apparently referable to *Hydractinia*, Van Beneden, by Loven§ in a *Syncoryne*, and by Dujardin‖ in another species of *Syncoryne*, whose medusoid was seen to produce ova in the thickness of the stomach-walls. The less perfectly developed medusoids which remain attached to the mouth of the capsule in *Campanularia* have also been observed to contain ova as noted by Loven¶; while the previous observation of Lister**, who saw multitudes of moving bodies escaping from these medusoids, is possibly also an instance of a similar phenomenon, though the form of the moving bodies as described by Lister leaves a doubt of their being truly embryos; they are probably spermatozoa, and then Lister's medusoids would be males; a view supported by Shultz's†† recent observation, that these attached medusoids of *Campanularia* sometimes contain spermatozoa instead of ova.

Now in *Cordylophora* I have never witnessed the production of medusoids, but I think nevertheless that it will be easily seen that the reproductive capsules of this genus have essentially the same organization as the medusoids of the marine genera. These capsules in an advanced stage exhibit but little of a medusoid conformation, but at an earlier period of their development, as we have just seen, they present us with an organized sac having a hollow central and fleshy column projecting into it below, and furnished with a system of branched canals which are developed on its walls, and communicate at their origin with the cavity of the central column. In this structure we can have no hesitation in recognising a true medusoid type; the organized cellular sac is homologous with the disc of the Medusa, the central fleshy column will represent the peduncle or proboscidiform stomach, while the system of branched tubes will correspond to the gastro-vascular canals‡‡. It is true that neither

* See Krohn in Müller's Archiv. † Vide *infra*, p. 378. ‡ Wagner in the Isis, 1833. Heft iii.

§ Loven in Wirgmann's Archiv, 1837.

‖ Dujardin in Ann. des Sci. Nat. 3me sér. tom. iv. 1843. ¶ Loven, *l. c.*

** Lister in Philosophical Transactions, 1844. †† Shultz in Muller's Archiv, 1851.

‡‡ The ramified tubes of the reproductive capsules must, I think, be also assumed as homologous with the tentacles of the terminal polypes. The tentacles of a hydroid polype will thus find their homologues in the gastrovascular canals of a medusa.

disc nor stomach presents any obvious opening in the capsules of the polype, but this is merely a particular and non-essential modification of structure adapted to the function which more especially devolves on them, and it must be recollected that being in direct communication with the common cavity of the parent stem, they are in no need of the admission of nutritive matter from without. Neither does the ramified condition of the tubes, nor the absence of the marginal canal, invalidate the view here taken, for we know that the gastro-vascular canals in certain true Medusæ are ramified, while the closed condition of the disc brings with it the absence of the marginal canal*.

There is a difficulty in the determination of the exact part of the reproductive capsules, which is immediately concerned in the production of the ova or spermatozoa. As soon as ever these bodies become apparent, they are seen to occupy the entire space between the diverticulum and the walls of the inner vesicle. The observation already mentioned, as made by Dujardin on a *Syncoryne*, in whose free medusoid the ova were produced in the substance of the stomach-walls, would render it probable that in *Cordylophora* also they had their origin in the part homologous, namely, the central diverticulum, and that from this they became liberated at a very early age to undergo subsequent development in the cavity of the capsule; and this view is further supported by what I have myself observed in the fixed egg-bearing organs of *Sertularia argentea*, to be presently referred to.

With the reproductive capsules of *Cordylophora* we may now compare the fixed ovigerous sacs of the marine genera. It will be seen that these also possess a true medusoid structure, though this is perhaps not quite so easily made out in every case as in *Cordylophora*. In the marine *Tubulariadæ*, the sacs in question consist of a closed vesicle composed of cells, and surrounded externally by a delicate structureless capsule; into its bottom projects a large, hollow, cylindrical and contractile organ, whose cavity is in connection with that of the supporting polype, and round which the ova are developed. Now this central organ corresponds exactly to the central diverticulum in the reproductive capsules of *Cordylophora*, and plainly represents the proboscidiform stomach of a Medusa, while the cellular sac is homologous with the disc. It is true there is nothing here which evidently represents the gastro-vascular canals of the Medusæ and the ramified tubes in the capsules of *Cordylophora*, but there is yet quite sufficient essential resemblance to justify us in referring the reproductive capsules of the marine *Tubulariadæ* to the type of medusoid structure, to which we have just seen that those of *Cordylophora* admit of a close comparison.

In the *Campanulariæ*, besides the bodies of obvious medusoid structure which have been noticed by so many observers to escape from the capsule and then swim freely away, and those of less completely developed, though still obvious medusoid structure which remain attached at the mouth of the capsule, others are commonly seen in the

* The medusoids of the marine *Tubulariadæ* are, during the early stages of their development, contained in a delicate external sac which afterwards disappears. This is the homologue of the external investment of the reproductive capsules of *Cordylophora* and the marine *Tubulariadæ*.

interior of the capsule clustering round the central axis, and described by various naturalists as eggs. In these so-called eggs, however, there may be detected a medusoid structure quite as manifest as in the ovigerous sacs of the marine *Tubulariadæ*; each of them is in fact a fixed Medusa developing within it true ova which possess the germinal vesicle and germinal spot, and present the phenomenon of yelk-cleavage, and after a time escape as locomotive ciliated embryos. It is probable however that in some cases the production of spermatozoa instead of ova is the true office of these bodies.

In *Sertularia argentea* I have found the axis developed in the interior of the capsule into a body of very obvious medusoid conformation. In this body the medusoid structure has experienced an advance over that of the reproductive sacs of the *Tubulariadæ*; both mouth and disc are open, and four unbranched gastro-vascular canals are present, but it is permanently fixed, no marginal tentacula are developed, nor could I observe the least motion of the disc corresponding to the systole and diastole of the medusoids of the *Tubulariadæ* and the *Campanulariæ*. It is supported on a short stem, which springs from the bottom of the capsule, and is directly continuous with its stomach. In this stem the ova originate and appear to escape from it into the stomach of the medusoid, to be discharged from the mouth into an external delicate vesicle, where they are retained during subsequent stages of their development.

If the views now taken be correct, the reproductive capsules of *Cordylophora*, and those of the marine *Tubulariadæ*, as well as the parts immediately concerned in the sexual reproduction of the *Sertulariadæ*, must be regarded as distinct zooids*, presenting a more or less degraded type of medusoid structure†.

On investigations into the embryology of the lower polypes, an important influence was exercised by Ehrenberg's determination of the signification of the ovigerous capsules of *Campanularia* and *Coryne*. The existence of these capsules was, as is well known, supposed by the celebrated Berlin micrographist‡ to indicate a distinction of sex in the polypes of a zoophyte, the ordinary terminal polypes being considered by him as non-sexual, while the ovigerous capsules were maintained to be female polypes. This view was ardently adopted by Loven§, who, observing

* The introduction of the term *zooid* into the language of zoology is of very recent date. This term is intended to indicate each of the distinct organisms which, with various degrees of independence, express when taken together the total result of the development of a single *ovum*. It is a valuable addition to our terminology, enabling us to avoid the ambiguous sense which attaches itself to the word *individual* when this word is used in its biological signification as the logical element of a *species*. See Huxley, observations on Salpa, &c., in Philosophical Transactions, 1851, and Lecture on Animal Individuality, Ann. Nat. Hist. June 1852. See also Carpenter, Princ. Gen. and Comp. Physiol. p. 906.

† Our knowledge of the egg-bearing and spermatozoa-bearing bodies of *Hydra* is not yet sufficiently accurate to enable us to decide with certainty how far the reproductive organization of this animal should be included in the same type with that of *Cordylophora* and the marine hydroid zoophytes. I believe that indications of a medusoid type may be here also witnessed, but further observations are necessary for the complete elucidation of this point.

‡ Corallenthiere des Rothen Meeres.

§ Wiegmann's Archiv, 1837.

some of the free medusoids of a *Syncoryne* to be filled with ova, supposed these medusoids to possess the same signification as that which Ehrenberg had attributed to the fixed capsules. Steenstrup, adopting with some slight modification the same view, has found in it one of the most striking of the facts which he has so skilfully collated in his essay on "The Alternations of Generations;" while Van Beneden*, in two elaborate memoirs full of beautiful and accurate observations, has strenuously opposed it, maintaining that the ovigerous fixed capsules are merely organs, "ovisacs," while the free medusoids are larvæ destined to undergo a series of transformations during a process of development into the form of the parent zoophyte. M. Van Beneden, however, though he has had the medusoids long under his eyes, has never witnessed an actual development into the form of the parent polype, and the changes which he has figured as occurring in them do not appear to show any tendency to such a transformation. I have had the medusoids of a *Syncoryne* for more than a fortnight under constant inspection without perceiving the slightest change towards the form of the adult polype; at the end of this period they perished.

Dujardin† opposes the views both of Ehrenberg and Van Beneden, and taking the medusoids for true *Medusæ*, and viewing them even as distinct genera of *Acalepha*, he generalizes the observed instances of their production, and maintains that the claviform polypes are universally only inferior phases of the development of the *Acalepha*; while he considers the bodies produced in the fixed sacs only as "*bulbillæ*," by which he understands gemmæ, which become detached at an early stage, and are then capable of an independent development. The strong resemblance however between true *Medusæ* and the medusoids produced by a process of gemmation from polypes does not afford sufficient grounds for maintaining that all *Medusæ* are only advanced stages of tubularian zoophytes and of *Campanulariæ*, or that all these zoophytes are only earlier phases in the development of *Medusæ*; and M. Dujardin's generalization, from which would result the complete abolition of the *Campanulariæ* and tubularian zoophytes as distinct groups of the animal kingdom, is certainly destitute of sufficient foundation, while the bodies contained in the fixed sacs are undoubtedly not *gemmæ* but *ova*, as their structure and the whole history of their development must render evident. I cannot think, then, that either Van Beneden or Dujardin has succeeded in overthrowing the theory of Ehrenberg.

The conclusion to which the facts sought to be demonstrated in the present paper would seem to lead, is that in the tubularian zoophytes there exist three kinds of zooids, all produced by a process of gemmation from an original stolon, which is itself the immediate product of a true ovum. These are,—1. the ordinary terminal

* Mém. sur les Campanulaires, and Recherches sur l'Embryogenie des Tubulaires, Nouv. Mém. de l'Acad. Roy. de Bruxelles, t. vii.

† Dujardin, Sur le Développement des Méduses et des Polypes Hydraires, Ann. des Sci. Nat. 3me sér. t. iv. 1845.

zooids, on which devolves more especially the function of the reception, digestion, and elaboration of the nutrient material; 2. the fixed reproductive zooids, in which every function is rendered subordinate to the production and development of the ova or spermatozoa, and in which a medusoid structure is more or less disguised; and 3. the free medusiform zooids, in which a complete medusoid structure is obvious. The office of these last is less exactly determined than that of either of the others; sexual reproduction is probably, as in the fixed reproductive zooids, their characteristic function; for as we have already seen, they have in several instances been found to contain ova; yet in the greater number of cases no ova have been detected in them, and observations are still wanting to enable us to decide whether at one period or another of their existence a true generative office is not, under favourable circumstances, always performed by them. It is certain that the other functions are not here so completely subordinated to reproduction as in the fixed sacs, for since the medusoid has become entirely detached from the parent, a more elaborate organization is necessary to enable it to maintain an independent existence. Uniting the functions of reproduction with extensive locomotive powers, it is probably destined to carry the ova to a distance from the parent stock, and thus provide for the dispersion of the species, for the motion of the ciliated embryo is too slow to be of any use in this respect. In the *Campanulariæ*, besides these three forms of zooids there exists a fourth, in which the medusoid structure is also obvious, though not so completely developed as in the free medusoids: the bodies belonging to this group remain permanently fixed at the mouth of the capsule; their function is that of true sexual generation, and after giving birth to ova or spermatozoa they wither away. The ovigerous sacs of degraded medusoid structure concealed in the interior of the capsules of *Campanularia* are quite different from these, and belong to the second kind of zooid enumerated above. In *Sertularia* and its immediate allies no locomotive zooids have as yet been found, and the generative zooids of *Sertularia argentea* already described admit of comparison rather with the fixed medusoids attached to the mouth of the capsule in *Campanularia*, than with the ovigerous sacs of the *Tubulariadæ*.

It would seem then that Ehrenberg struck upon the true determination of the ovigerous sacs in *Coryne* when he called them "female polypes," though he erred when, supposing the "ovarian vesicles" of *Campanularia* to be the homologues of the ovigerous sacs of *Coryne*, he called these vesicles also by the same name; the true homologues of the reproductive sacs of the *Tubulariadæ* being, not the external capsules of the *Campanulariæ*, but fixed bodies with a disguised medusoid structure contained in the interior of these capsules*.

The present state of our knowledge of the Tubularian and Sertularian polypes would seem to justify the generalization, that for the production of ova in these

* Krohn has already pointed out that the capsules in *Campanularia* have a signification different from that of the ovigerous sacs of *Coryne*, *l. c.*

polypes a more or less perfectly developed medusoid organization is required, this organization being demonstrable not only in the free medusoids themselves, which are occasionally observed to give origin to ova, but in those reproductive capsules and imperfectly developed medusoids which remain permanently adherent to the parent.

Description of the Plates.

PLATE XXV.

Fig. 1. *Cordylophora lacustris*, attached to a dead valve of *Anodon cygneus*. Natural size.

Fig. 2. A branch magnified with the polypes in various states of expansion, and with the reproductive capsules more or less developed.

Fig. 3. Longitudinal section of polype to show the details of its structure.

- *a*. Ectoderm.
- *b*. Polypary.
- *c*. Processes from the ectoderm attached to inner surface of the polypary.
- *d*. Endoderm.
- *e*. Mouth.
- *f*. Post-buccal cavity.
- *g*. Stomach.
- *h*. Common canal of the cœnosarc.
- *i*. Muscles.

PLATE XXVI.

Fig. 4. Portion of the walls of the stomach more highly magnified.

- *a*. Ectoderm, its cells containing thread-cells.
- *b*. Endoderm composed of elongated cells, with true secreting cells in their interior.
- *c*. Secreting cells with evident nucleus.
- *d*. Secreting cells with the nucleus obscured by the opake contents.
- *e*. Granular mass.
- *f*. Muscles.

Fig. 5. Thread-cells previous to the exsertion of the filament.

Fig. 6. Thread-cells after the exsertion of the filament.

Fig. 7. A group of cells liberated under pressure from the ectoderm. Some contain a single thread-cell, others a nucleus-like body, probably an undeveloped thread-cell. They appear to have been originally contained as secondary cells within the proper cells of the ectoderm.

Fig. 8. Cells liberated by pressure from the endoderm of the stomach. They are filled with smaller cells, which contain coloured granules, and seem to have been originally contained within the proper cells of the endoderm.
a. Nuclei of contained cells distinct.
b. Nuclei concealed by the coloured granules.

Fig. 9. Portion of a tentacle near its root, to show its structure.
a. Ectoderm with thread-cells.
b. Endoderm, its cells containing coloured granules.
c. Muscular fibres.

Fig. 10. Cells containing secondary cells with granular contents from the endoderm, when covered by the polypary.

*** In figs. 11–14 the same letters are used to indicate corresponding portions of structure, viz. *a*. Diverticulum from the cœnosarc. *b*. External investment of the reproductive capsule continued from the polypary. *c*. Cellular sac. *d*. Ramified canals. *e*. Structureless sac secreted on the outside of the cellular sac.

Fig. 11. Reproductive capsule—very early stage.

Fig. 12. The same—more advanced stage; the cellular sac and ramified canals are now visible.

Fig. 13. Ideal longitudinal section of a reproductive capsule of about the same period as fig. 12; showing the relation of the ramified canals to the diverticulum and cellular sac, and the presence of a delicate structureless sac which has been secreted on the outside of the cellular sac.

Fig. 14. Reproductive capsule in a more advanced stage; the ova are now visible in its interior.

Fig. 15. Reproductive capsule still further advanced; the cellular sac and ramified canals have disappeared, and the ova are now seen lying upon the extremity of the diverticulum, and immediately included within the structureless sac which had been formed upon the outside of the cellular sac.

Fig. 16. A more magnified view of an ovum from fig. 15; the germinal vesicle has disappeared, and the segmentation of the vitellus has converted the ovum into a mulberry-like mass.

Fig. 17. A capsule still further advanced; the mulberry-like condition has disappeared, and the ova have begun to elongate themselves.

Fig. 18. A capsule in a still more advanced stage; the ova have become still more elongated, and they now present a swarming motion in the interior of the capsule.

Fig. 19. Termination of the intra-capsular life of the embryo; the enclosing sacs have become ruptured, and the embryos are escaping in the form of free ciliated infusoria.

Fig. 20. Embryo just after its escape from the capsule, more highly magnified; it presents an internal closed cavity with ectoderm and endoderm.

Fig. 21. The same, showing the pyriform figure which it frequently assumes.

Fig. 22. Embryo subsequently to its locomotive stage; the cilia have disappeared, and it has become permanently fixed by one extremity.

Fig. 23. Further progress in the development of the young polype; the tentacula have begun to bud forth, and the stem is already surrounded by a delicate polypary.

Fig. 24. Male capsule with its contents.

a. Capsule with the contents escaping under slight pressure.

b. Caudate cells liberated from the capsule.

c. Spermatozoa.

XVI. *On the Change of Refrangibility of Light.*—No. II. *By* G. G. Stokes, *M.A., F.R.S., Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

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THE chief object of the present communication is to describe a mode of observation, which occurred to me after the publication of my former paper, which is so convenient, and at the same time so delicate, as to supersede for many purposes methods requiring the use of sun-light. On account of the easiness of the new method, the cheapness of the small quantity of apparatus required, and above all, on account of its rendering the observer independent of the state of the weather, it might be immediately employed by chemists in discriminating between different substances.

I have taken the present opportunity of mentioning some other matters connected with the subject of these researches. The articles are numbered in continuation of those of the former paper.

Method of observing by the use of Absorbing Media.

241. Conceive that we had the power of producing at will media which should be perfectly opake with regard to rays belonging to any desired regions of the spectrum, from the extreme red to the most refrangible invisible rays, and perfectly transparent with regard to the remainder. Imagine two such media prepared, of which the second was opake with regard to those rays of the *visible* spectrum with regard to which the first was transparent, and *vice versâ*. It is clear that if both media were held in front of the eye no light would be perceived. The same would still be the case if the first medium were removed from the eye, and placed so as to intercept all the rays which fell on certain objects, which were then viewed through the second, provided the objects did nothing more than reflect, refract, scatter, or absorb the incident rays. But if any of the objects had the property of emitting rays of one refrangibility under the influence of rays of another, it might happen that some of the rays so emitted were capable of passing through the second medium, in which case the object would appear luminous in a dark field.

242. Let us consider now how the media must be arranged so as to bring out to the utmost the sensibility of a given substance. To take a particular instance, suppose the substance to be glass coloured by uranium. In this case the sensibility of the medium begins, with almost absolute abruptness, near the fixed line *b* of Fraunhofer, and continues from thence onwards. The dispersed light has the same,

or at least almost rigorously the same, composition throughout, and consists exclusively of rays less refrangible than *b*. Consequently, we should have to prepare a first medium which was opake with regard to the visible rays less refrangible than *b*, and transparent with respect to the rays, whether visible or invisible, more refrangible, and a second medium complementary to the former in the manner described in the preceding article. If the pair of media were still strictly complementary in this manner, but the point of the spectrum at which the transparency of the first medium began and that of the second ended were situated at some distance from *b*, the sensibility of the glass would be exhibited as before, only the maximum effect would not be produced, on account of the absorption of a portion either of the active or of the dispersed rays, according as the point in question was situated above or below *b*.

Now, although the commencement of the sensibility of canary glass is unusually abrupt, it generally happens that the sensibility of a medium, or at least the main part of it, comes on with great rapidity, and lasts throughout the rest of the spectrum, though frequently it is most considerable in a region extending not very greatly beyond the point where it commenced. In those cases in which the dispersion of different tints commenced at two or three different places in the spectrum, I have almost always had evidence of the independent presence of different sensitive principles, to which the observed effects were respectively due.

Hence, if we could prepare absorbing media at pleasure, we should get ready for general use in these observations a few pairs of media complementary in the particular manner already described, but having the points of the spectrum at which the transparency of the first medium commenced and that of the second ended different in different pairs, situated say in the yellow for one pair, in the blue for another, in the extreme violet for a third.

243. It is not of course possible to prepare media in this manner at pleasure, and all we can do is to select from among those which occur in nature. Nevertheless it is useful, as a guide in the selection, to consider what constitutes the ideal perfection of absorbing media for this particular purpose. But before proceeding to mention the media which I have found convenient, I will describe the arrangement which I have adopted for admitting the light.

A hole was cut in the window-shutter of a darkened room, and through this the light of the clouds and external objects entered in all directions. The diameter of the hole was four inches, and it might perhaps have been still larger with advantage. A small shelf, blackened on the top, which could be screwed on to the shutter immediately underneath the hole, served to support the objects to be examined, as well as the first absorbing medium. This, with a few coloured glasses, forms all the apparatus which it is absolutely necessary to employ, though for the sake of some experiments it is well to be provided also with a small tablet of white porcelain, and an ordinary prism, and likewise with one or two vessels for holding fluids.

244. In the observation, the first medium is placed resting on the shelf so as to cover the hole; the object is placed on the shelf immediately in front of the hole; the second medium is held anywhere between the eyes and the object. As it is not possible to obtain media which are strictly complementary, it will happen that a certain quantity of light is capable of passing through both media. This might no doubt be greatly reduced by increasing the absorbing power of the media, but it is by no means advisable to do so to any great extent, because it is important that the second medium should transmit as many as possible of the rays which are of such refrangibilities as to be stopped by the first. Accordingly, it might sometimes be doubtful whether the illumination perceived on the object were due merely to scattered light which had passed through both media, or to really "degraded" light*. To remove all doubt, it is generally sufficient to transfer the second medium from before the eyes to the front of the hole. The light merely scattered by the object will necessarily be the same as before, if the room be free from stray light; and even if there be a little stray light, the illumination, so far as it is concerned, will be increased instead of diminished; whereas if the illumination previously observed were due to fluorescence, and the media were properly chosen, the object which before was luminous will now be comparatively dark.

Sometimes, in the case of substances which have only a low degree of sensibility, it is better to leave the second medium in front of the eyes, and use a third medium, which is held alternately in the path of the rays incident on the object and between the object and the eyes. Such a medium, though not at all necessary, may be used

* This term, which was suggested to me by my friend Prof. Thomson, appears to me highly significant. The expression *degradation of light* might be substituted with advantage for *true internal dispersion* to designate the general phenomenon; but it is perhaps a little too wide in its signification, and might be taken to include phosphorescence (if indeed in this case the refrangibility be really always lowered), as well as the emission of non-luminous radiant heat by a body which had been exposed to the red rays of the spectrum. As to the term *internal dispersion*, though I employed it, following Sir David Brewster, I confess I never liked it. It seems especially awkward when applied to a washed paper or dyed cloth; it was adopted at a time when the phenomenon was confounded with opalescence; and, so far as it implies theoretical notions at all, it seems rather to point to a theory now no longer tenable: I allude to the theory of suspended particles. Indeed, this theory, as it seems to me, ceased to be tenable as soon as Sir John Herschel had discovered the peculiar analysis of light connected with epipolic dispersion, and Sir David Brewster had connected the phenomenon with internal dispersion, so far at least as the common appearance of a continuous and coloured dispersed beam formed a connexion. The expression *dependent emission* is awkward, but would be significant, because the light is emitted in the manner of self-luminous bodies, but only in dependence upon the active rays, and so long as the body is under their influence. In this respect the phenomenon differs notably from phosphorescence. It is quite conceivable that a continuous transition may hereafter be traced by experiment from the one phenomenon to the other, but no such transition has yet been traced, nor is it by any means certain that the phenomena are not radically distinct. On this account it would, I conceive, be highly objectionable to call true internal dispersion *phosphorescence*. In my former paper I suggested the term *fluorescence*, to denote the general appearance of a solution of sulphate of quinine and similar media. I have been encouraged to give this expression a wider signification, and henceforth, instead of true internal dispersion, I intend to use the term fluorescence, which is a single word not implying the adoption of any theory.

also in the case of highly sensitive substances, for the sake of varying the experiment and rendering the result more striking.

As it will be convenient to have names for the media fulfilling these different offices, I will call the first medium, or that with which the hole is covered, the *principal absorbent*, the second medium the *complementary absorbent*, and the third medium, when such is employed, the *transfer medium*. For the transfer medium we may choose a medium of the same nature as the complementary absorbent, but paler. This is perhaps the best kind to employ in the methodical examination of various substances; but if the object of the observer be merely to illustrate the phenomena of the change of refrangibility of light, he may vary the experiments by using other media.

245. I have hitherto spoken only of the increase of illumination due to the sensibility of the substance under examination. But independently of illumination, the colour of the emitted light affords in most cases a ready means of detecting fluorescence. Thus, suppose the principal absorbent to transmit no visible rays but deep blue and violet, and the substance examined to appear, when viewed through the complementary absorbent, of a bright orange colour. Since no combination of the rays transmitted by the principal absorbent can make an orange, we may instantly conclude that the substance is sensitive. However, I do not consider it safe, at least for a beginner, to trust very much to *absolute* colour, for few who have not been used to optical experiments can be aware to what an extent the eye under certain circumstances is liable to be deceived. The *relative* colour of two objects seen at the same time under similar circumstances may usually be judged of safely enough; that is, of two such colours it may be possible to say with certainty that one inclines more to blue or to red than the other. Of course in many cases the change of colour is so great that there can be no mistake; still I think it a safe rule for a person employing these modes of observation without having been previously used to optical experiments, to require some other proof of a change of refrangibility than merely a change of colour. Experience will soon show what appearances may safely be relied on.

246. If it be desired to view the object isolated as much as possible, it may be placed directly on the shelf, or better still, on black velvet. But it is generally preferable to have for comparison a standard object which reflects freely the visible rays, of whatever kind, incident upon it, and does not possess any sensible degree of fluorescence. It is in this way that the white porcelain tablet is useful; and in observing, I generally place the tablet on the shelf and the object on it. A white plate would answer, but a tablet is better, on account both of its shape and of the comparative dullness of its surface. It is true that the tablet used exhibited a very sensible amount of fluorescence when examined in a linear spectrum formed by a quartz train; still the effect was so small, and so much of it was due to those highly refrangible rays which are stopped by glass, that for practical purposes the tablet may be regarded as insensible. However, an observer is not obliged either to assume that all tablets are alike, or to apply to the particular tablet which he proposes to use, methods of

observation requiring the use of apparatus which he is not supposed to possess. The methods of observation described in the present paper are complete in themselves; the observer has it in his power to test for himself the tablet he proposes to employ; and he is bound to do so before taking it for a standard of comparison. It may easily be tested by means of a prism, as will be explained presently.

247. The following are the combinations of media which I have chiefly employed:—

First combination.—In this combination the principal absorbent is a glass coloured deep violet by manganese with a little cobalt, or else a glass coloured deep purple by manganese alone, combined with a rather pale blue glass coloured by cobalt, or with a deeper blue glass in case the day be bright. It is very easy to tell by means of a prism, with candle-light, whether a purple glass contains any sensible quantity of cobalt, on account of the very peculiar mode of absorption which is characteristic of this metal. In the examination of substances by this combination no complementary absorbent is usually required; but if it be wished to employ one, a pale yellow glass, of the kind mentioned in connexion with the next combination, may be used.

Second combination.—In this case the principal absorbent is a solution of the ammoniaco-sulphate of copper, employed in such thickness as to give a deep blue. In my experiments the fluid was contained in a cell with parallel sides of glass, which was closed at the top for greater convenience; but a very broad flat bottle would answer as well, because in the case of the principal absorbent the regularity of refraction of rays across it is of no consequence. Such a bottle, however, would have to be ordered expressly. The complementary absorbent in this combination is a yellow glass coloured by silver, and slightly overburnt. These glasses, as commonly prepared, are opake with regard to most of the violet, but become transparent again with regard to the invisible rays beyond; and, in the case of a pale glass, the commencing transparency in the extreme violet may even be perceived by means of light received directly into the eye. I have got a glass of a pretty deep orange-yellow colour, which is more transparent than common window-glass with regard to rays of such high refrangibility as to be situated near the end of the region of the solar spectrum which it requires a quartz train to show. But when too much heat is used in the preparation, the glass acquires, on the interior of the coloured face, a delicate blue appearance, having a good deal of the aspect of a solution of sulphate of quinine, though it has in reality nothing to do with fluorescence; and in this state the glass is nearly opake with regard to the invisible rays of the solar spectrum beyond the violet, though it still transmits a few among those which are nearly the most refrangible. Of course, if the complementary absorbent were always left in its position between the eyes and the object, its transparency or opacity with regard to invisible rays would be a matter of indifference; but as it is desirable that its transference from that position to the front of the hole should produce as much difference as possible, it is important that it should be opake, or nearly so, with regard to the ultra-violet rays transmitted by the principal absorbent. Hence one of these slightly

over-burnt glasses should be selected for the present purpose, and such are very commonly met with.

Third combination.—In this case everything is the same as in the preceding combination, except that the fluid is replaced by a glass of a pretty deep blue, coloured by cobalt.

Fourth combination.—In this the principal absorbent is a solution of nitrate of copper, and the complementary absorbent a light red or deep orange glass.

248. In the first combination the darkness is tolerably complete without the use of any complementary absorbent, since no visible rays are transmitted except violet and some extreme red. The latter are no inconvenience, but rather help to set off the tint of the light due to fluorescence. This is, I think, the best combination to employ when the fluorescent light is blue, or at least deep blue; because in that case much of the light is lost by absorption in the yellow glass employed in the second combination. It has the advantage, too, of allowing the fluorescent light to enter the eye without being modified by absorption. Nevertheless no correct estimate can be formed of the absolute colour of the fluorescent light without making very great allowance for the effects of contrast, especially when the body, instead of being isolated as far as possible, is placed on the porcelain tablet.

The second combination is on the whole the most powerful. The media in this case make a very fair approach towards the ideal perfection explained in Art. 242. The darkness is so far complete, or else may easily be made so by increasing a little the strength of either absorbent, that if the tablet be written on with ink, and placed on the shelf between the media, the writing cannot be read. It forms a striking experiment, after having treated the tablet in this manner, to introduce between the media a piece of canary glass or a similar medium. The glass is not only luminous itself, but it emits so much light as to illuminate the whole tablet, so that the writing is instantly visible. In those cases in which the fluorescent light is yellow, orange, or red, it is shown a good deal more strongly by this combination than by the preceding.

The third combination is applicable to the same cases as the second. The blue glass answers extremely well, but is not quite so good as the blue fluid. The darkness is less complete, on account of the red and yellow transmitted by the glass. Nevertheless this combination is sometimes useful in observing with a prism, and at any rate it may very well be employed by a person who does not happen to have a vessel of the proper shape for holding the fluid.

In the second and third combinations the point of the spectrum at which the transparency of the principal absorbent begins, and that of the complementary absorbent ends, or rather the point which most nearly possesses this character, is situated in the blue. Thinking that the fluorescence of those substances which emit light of low refrangibility might be better brought out if this point were situated lower down in the spectrum, I tried the fourth combination. In this case the media have very fairly the required complementary character; the darkness is pretty com-

plete, and the fluorescence of scarlet cloth and similar substances is very well exhibited. However, the effect in these cases is shown so well by the second combination, that, except it be for the sake of varying the experiment, I do not think it worth while to employ the fourth combination, more especially as it has the disadvantage of leaving the observer in doubt whether the red or orange light perceived constitutes the whole of the fluorescent light, or only that part of it which alone has been able to get through the complementary absorbent.

249. The mode of observation may be altered in various ways which afford pleasing illustrations of the theory, though in the regular examination of a set of substances it is best to proceed in a more methodical manner. Thus, if nothing but a violet or blue glass or a blue fluid be used as a principal absorbent, and the substances under examination be highly sensitive, their appearance will be remarkably changed if the coloured medium be transferred from before the hole to before the eyes. Again, if the complementary absorbent be made to exchange places with the principal absorbent, the result will be similar, although the very same media are merely interposed in different parts of the compound path of the light from the clouds to the eye. If a transfer medium be employed, and it be, as has hitherto been supposed, of the same general nature as the complementary absorbent, it will not produce much effect when it is interposed between the object and the eyes, but when it is placed in the path of the rays incident on the object, the fluorescent light will be nearly if not entirely cut off. If, however, we take for a transfer medium a glass or fluid having the same general character as the principal absorbent, the effect will be just the reverse. This is strikingly shown in the case of a substance, which, like scarlet cloth, emits a red fluorescent light, by taking for a transfer medium a solution of nitrate of copper, and in the case of turmeric paper or yellow uranite, by taking the same solution, or else a blue glass. In the case of the two substances last mentioned, if we take for a transfer medium a red solution of mineral chameleon, diluted so as to be merely pink, the intensity of the light emitted will, under certain conditions, be not much different in the two positions of the medium, because a portion of the active rays in one position and a portion of the degraded rays in the other will be absorbed; but the colour of the portion of the emitted light which reaches the eye will be altogether different in the two positions of the transfer medium.

Mode of observing by means of a Prism.

250. In this method no absorbing medium is required except the principal absorbent. The white tablet being laid on the shelf, a slit is first held in such a position as to be seen projected against the sky, and the light thus coming directly into the eye, after having passed through the principal absorbent, is analysed by a prism held in the other hand. The slit is now held so that the tablet is seen through it, and the light coming from the tablet is analysed. It will be found that the spectrum seen in the first instance is faithfully reproduced, being merely less lumi-

nous, as must necessarily happen. At least, this was the case in those tablets which I have examined; and in this way each observer ought to test for himself the tablet he proposes to employ. After having been thus tested, the tablet may be used as a standard of comparison.

Suppose now that it is wished to examine a slip of turmeric paper, or a riband, or other similar object. The object is laid on the tablet, and the slit held immediately in front of it, in such a manner that one part, suppose the central portion, of the slit is seen projected on the object, and the remainder on the tablet. The light coming through the slit is then analysed by the prism, and the fluorescence, if any, of the object is indicated by light appearing in those regions of the spectrum in which, in the case of the light scattered by the tablet, there is nothing but darkness.

Occasionally in these observations a blue glass is preferable to a solution of the ammoniaco-sulphate of copper, because the extreme red and the greenish yellow bands transmitted by the glass, while too faint to interfere with the fluorescent light, are useful as points of reference.

Fig. 1.

Fig. 2.

251. The general appearance of the spectrum in this mode of observation may be gathered from the accompanying figures, of which the first represents turmeric paper seen under the blue glass, and the second represents a mass of crystals of nitrate of uranium seen under the copper solution. In fig. 1, RR′, YY′ are the red and yellow bands transmitted by the glass, which are seen equally in the light scattered by the tablet and that scattered by the paper. BVB′V′ is the blue and violet light transmitted by the glass. Of this a considerable portion, especially in the more refrangible part, is absorbed by the turmeric paper, which on the other hand emits a quantity of red, yellow, and green light, not found among the incident rays. Fig. 2 sufficiently explains itself. In this case the fluorescent light is decomposed by the prism into bright bands, of which six may be readily made out. No blue or violet light enters the eye from the part of the slit which is seen projected on the mass of crystals, except where a crystalline face happens to be situated in such a position as to reflect the light of the sky into the eye, as represented in the figure. In the case of a substance so highly sensitive as nitrate of uranium, and which does not, like a slip of paper, lie flat on the tablet, the spectrum of the fluorescent light in reality extends, at least on the side next the window, though with less intensity, to some distance beyond the part of the slit which corresponds to the object, because the tablet is lighted up by the rays emitted by the object; but this is not represented in the figure.

252. The mode of using the prism just explained is that by which the phenomenon

of the change of refrangibility is most strikingly illustrated; but in the actual examination of substances the chief use of the prism is to determine, in the case of substances which are sufficiently sensitive to admit with advantage of such a mode of observation, the composition of the fluorescent light. For this purpose it is often better to isolate the object by placing it on black velvet. This is especially the case with very minute crystals, or other objects, which are best placed on black velvet, and viewed through the prism as a whole, no slit being required.

Examples of the application of the preceding methods of observation.

253. The peculiar properties of paper washed with tincture of turmeric or stramonium seeds, of yellow uranite, and other highly sensitive substances, come out in a remarkable manner under the modes of examination described in this paper. I need not say that such is the case with solutions of sulphate of quinine, or horse-chestnut bark, or other clear and highly sensitive media, seeing that in this case the appearance due to fluorescence is obvious to common observation. If a piece of horse-chestnut bark be put to float in a glass of water close to the hole covered by the principal absorbent, the appearance of the descending streams of solution of esculine is very singular and beautiful. My present object is however rather to illustrate the power of these methods by their application to substances which stand much lower in the scale.

By the use of absorbing media alone, as well by a principal absorbent and a prism, I have been able to detect without difficulty the sensibility of white paper on a day of continuous clouds and rain. Even cotton wool, which stands very much lower in the scale, is shown by the use of absorbing media with ordinary daylight to be sensitive. In the case of such substances as bone, ivory, white leather, the white part of a quill, which stand much higher in the scale, the most inexperienced observer could hardly fail instantly to detect the fluorescence. All plates of colourless glass which I have examined, and other pieces which were of such a shape as to admit of being looked into edgeways to a considerable depth, were found by the second combination to be sensitive. Crystals of sulphate of quinine, which may be readily prepared from the disulphate of commerce, show their fluorescence extremely well by the first combination. These crystals are much less sensitive than their solution, and the light which they emit is of a much deeper blue. It must in reality be of a very deep blue colour; for it nearly matches the fluorescent light of fluor-spar, although when the crystals are viewed under the violet glass the tint in both cases appears comparatively pale, from contrast with the violet. A solution of nitrate of uranium on the other hand has only a low degree of sensibility compared with the crystals of that salt. If a drop of the solution be placed on the porcelain tablet when the hole is covered with the deep blue copper solution, it appears comparatively dark, because much more illumination is lost by the absorption of the indigo and violet than is gained by the fluorescence of the solution. But when the tablet is viewed through the complementary absorbent, the solution is seen to be more luminous than the tablet, and to emit yellow rays, which are not found in the incident light.

The reactions of quinine mentioned in my former paper (Arts. 205–208), may very conveniently be observed by means of drops of the solution placed on the tablet; and in this way it is possible to work in a perfectly satisfactory manner with excessively minute quantities of quinine. The statement there made, that the blue colour was *destroyed* by hydrochloric acid, &c., must be understood only with reference to the mode of observation there supposed to be adopted, which was sufficient for the object in view. When the solutions are examined in a pure spectrum formed by sunlight, or even by the method described in the present paper, it is seen that the blue colour is not *absolutely* destroyed by hydrochloric acid, and is even developed to a *slight* extent on the addition of hydrochloric acid to a previously alkaline solution. Still there is a broad distinction between the two classes of solutions, which is all that is required. I have since extended a good deal these results, and mean to pursue the subject further. Meanwhile I may be permitted to correct an error in Art. 205, relative to the effect of hydrocyanic acid, which was there stated to develope the blue colour. The experiment was made with the acid of commerce, containing a foreign acid, to which the effect was probably due.

Comparison of the relative advantages of different modes of observation.

254. At first sight it might have been supposed that daylight could never be more than a poor substitute for sunlight in any observations relating to fluorescence. Such, however, I consider to be by no means the case. In the first place, when sunlight is used it is made to enter a room in a definite direction; whereas in using absorbing media all the rays are employed whose directions lie within a solid angle having the object examined for vertex, and the hole for base. If we leave out the part of this solid angle which corresponds to trees or houses, the part which corresponds to sky will still be so large as to make up in a good measure for the superior brilliancy of the light of the sun. In the second place, stray light is much more perfectly excluded than when a beam of sunlight, containing rays of all kinds, is admitted into a room. When indeed the use of sunlight is combined with that of absorbing media, it is possible to detect very minute degrees of sensibility. Still for general purposes I consider the methods depending upon the use of absorbing media with ordinary daylight quite comparable with, if not equal to, those methods involving the use of sunlight which are applicable to opake bodies; I allude especially to the method of a linear spectrum. The peculiarities in the composition of the fluorescent light, when such exist, can be made out about equally well by both methods.

But when the substance to be investigated is a solution, or a clear solid of sufficient size to be examined as such, methods of observation can be put in practice with sunlight which surpass anything that can be done merely by the use of absorbing media. In consequence of the absence of stray light, which would otherwise dazzle the eye, an amount of concentration of the rays can be brought to bear on the object, which enables the observer to detect excessively minute degrees of sensibility. Thus, when the sun's light is condensed by a rather large lens, and made to

pass through a strong solution of the ammoniaco-sulphate of copper, the condensed beam of violet and invisible rays serves to detect fluorescence in almost all fluids. This, however, is no great advantage; the method is in fact too powerful; and the observer is left in doubt whether the effect perceived be due to the fluid deemed to be examined, or to some impurity which it contains in an amount otherwise perhaps inappreciable. The great advantage which sunlight observations possess in the examination of substances, which, however, is only applicable to clear media, is that they enable the observer to make out the distribution of activity in the incident spectrum. In some cases this constitutes the chief peculiarity in the mode of fluorescence of a particular substance; in other cases it enables the observer to see, as it were, independently of each other, different sensitive substances existing together in solution. Another advantage of sunlight, which applies equally to clear and to opake media, is that it enables the observer, with the assistance of a quartz train, to make out fluorescence which does not commence till that region of the spectrum which it requires a quartz train to show. But such cases are too rare to render this a point of much consequence. Of course there are observations such as those which relate to the fixed lines of the invisible rays, or to the determination of the absorbing action of a medium with regard to invisible rays of each degree of refrangibility in particular, which imperatively require sunlight: I am speaking at present only with reference to those observations of which the object is to investigate the mode of fluorescence of a particular substance.

As to the method of observation in which a prism is combined with a principal absorbent, its chief use is to determine, in the case of the more sensitive substances, the composition of the fluorescent light. It is not generally so convenient as the method which involves the use of absorbing media alone for determining which among a group of objects are sensitive, and which not, especially when the objects are minute.

255. Although the description of the mode of observing by means of absorbing media has run to some little length, the reader must not suppose that the observations are at all difficult. Of course observations of all kinds become more or less difficult when they are pushed to the extreme limit of refinement of which they are susceptible. But in the case of substances which are at all highly sensitive, and this comprises almost all the more interesting instances, the observations are extremely easy. I have spoken of a darkened room, which is certainly the most convenient when it can be had. But I have no doubt that an observer who could not procure such might easily arrange for himself a darkened box, which would answer the purpose. Indeed the fluorescence of highly sensitive substances, though they be opake, may be exhibited by means of absorbing media in broad daylight.

Platinocyanides.

256. In the Report of the Twentieth Meeting of the British Association (Edinburgh, 1850, Transactions of the Sections, p. 5), is a notice by Sir DAVID BREWSTER

of "The Optical Properties of the Cyanurets of Platinum and Magnesia, and of Barytes and Platinum," salts which he had received from M. HAIDINGER of Vienna. The notice is chiefly devoted to the properties of the reflected light; but with respect to the latter of the salts, Sir DAVID remarks that "it possesses the property of internal dispersion, the dispersed light being a *brilliant green*, while the transmitted light is *yellow*." Although the distinction between true internal dispersion and opalescence was not at the time understood, there could be little doubt from the nature of the case that the internal dispersion mentioned by Sir DAVID BREWSTER was, in fact, an instance of the former of these phenomena; but I could not try for want of a specimen of the salt. Some months ago I received from M. HAIDINGER a specimen of the first of the salts mentioned at the beginning of this paragraph, namely M. QUADRAT'S cyanide of platinum and magnesium, a salt of great optical interest on account of the remarkable metallic reflexion which it exhibits. On examining the salt, I was greatly interested by finding that it was highly sensitive, the fluorescent light being red. This induced me to form some of GMELIN'S platinocyanide of potassium, and I found that the blue light which this salt exhibits in certain aspects is, in fact, due to fluorescence, a property which the salt possesses in an eminent degree. Having afterwards received some of the same salt pure from Professor GREGORY, I applied it to the formation, on a small scale, of the platinocyanides of calcium, barium, strontium, and two or three others. The three salts last named, of which the second is that mentioned by Sir DAVID BREWSTER, are all eminently sensitive, the fluorescent light being of different shades of green. It is only in the solid state that the platinocyanides are sensitive; their solutions look like mere water. The precipitates which a solution of platinocyanide of potassium gives with salts of the heavy metals are, in most cases that I have yet observed, insensible. With a solution of pernitrate of mercury, however, a bright yellow precipitate is produced which is exceedingly sensitive, so as to look brighter than even yellow uranite. The light emitted is yellow. It forms a very striking experiment to place side by side on the tablet with the second combination a drop of a solution of platinocyanide of potassium, and another of pernitrate of mercury. The drops look like water on the dark field of view; but when they are mixed, a precipitate is produced which glows like a self-luminous body with a yellow light. The precipitate with nitrate of silver is also sensitive, though not in so high a degree. The platinocyanides are extremely interesting; first, as forming a third case, or rather class of cases, in which the property of fluorescence is attached to substances chemically isolated in a satisfactory manner (though I believe chemists are acquainted with a few other organic compounds to which the property belongs), the other two cases being salts of quinine and of peroxide of uranium; and secondly, as exhibiting a new and remarkable feature, which consists in the polarization of the fluorescent light. I content myself at present with this notice; the salts require a more extended study.

XVII. *On the Secular Variation of the Moon's Mean Motion.*

By J. C. Adams, *Esq.*, *M.A.*, *F.R.S.* &c.

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1. In treating a great problem of approximation, such as that presented to us by the investigation of the moon's motion, experience shows that nothing is more easy than to neglect, as insignificant, considerations which ultimately prove to be of the greatest importance. One instance of this occurs with reference to the secular acceleration of the moon's mean motion. Although this acceleration, and the diminution of the eccentricity of the earth's orbit, on which it depends, had been made known by observation as separate facts, yet many of the first geometers altogether failed to trace any connexion between them, and it was only after making repeated attempts to explain the phenomenon by other means, that Laplace himself succeeded in referring it to its true cause.

2. The accurate determination of the amount of the acceleration is a matter of very great importance. The effect of an error in any of the periodic inequalities upon the moon's place, is always confined within certain limits, and takes place alternately in opposite directions within very moderate intervals of time, whereas the effect of an error in the acceleration goes on increasing for an almost indefinite period, so that the calculation of the moon's place for a very distant epoch, such as that of the eclipse of Thales, may be seriously vitiated by it.

In the Mécanique Céleste, the approximation to the value of the acceleration is confined to the principal term, but in the theories of Damoiseau and Plana the developments are carried to an immense extent, particularly in the latter, where the multiplier of the change in the square of the eccentricity of the earth's orbit, which occurs in the expression of the secular acceleration, is developed to terms of the seventh order.

As these theories agree in principle, and only differ slightly in the numerical value which they assign to the acceleration, and as they passed under the examination of Laplace, with especial reference to this subject, it might be supposed that at most only some small numerical corrections would be required in order to obtain a very exact determination of the amount of this acceleration.

It has therefore not been without some surprise, that I have lately found that Laplace's explanation of the phenomenon in question is essentially incomplete, and that the numerical results of Damoiseau's and Plana's theories, with reference to it, consequently require to be very sensibly altered.

3. LAPLACE's explanation may be briefly stated as follows. He shows that the mean central disturbing force of the sun, by which the moon's gravity towards the earth is diminished, depends not only on the sun's mean distance, but also on the eccentricity of the earth's orbit. Now this eccentricity is at present, and for many ages has been, diminishing, while the mean distance remains unaltered. In consequence of this the mean disturbing force is also diminishing, and therefore the moon's gravity towards the earth at a given distance is, on the whole, increasing. Also, the area described in a given time by the moon about the earth is not affected by this alteration of the central force; whence it readily follows that the moon's mean distance from the earth will be diminished in the same ratio as the force at a given distance is increased, and that the mean angular motion will be increased in double the same ratio.

4. This is the main principle of LAPLACE's analytical method, in which he is followed by DAMOISEAU and PLANA; but it will be observed, that this reasoning supposes that the area described by the moon in a given time is not permanently altered, or in other words, that the tangential disturbing force produces no permanent effect. On examination, however, it will be found that this is not strictly true, and I will endeavour briefly to point out the manner in which the inequalities of the moon's motion are modified by a gradual change of the central disturbing force, so as to give rise to such an alteration of the areal velocity.

As an example, I will take the *Variation*, the most direct effect of the disturbing force.

In the ordinary theory, the orbit of the moon as affected by this inequality only, would be symmetrical with respect to the line of conjunction with the sun, and the areal velocity generated while the moon was moving from quadrature to syzygy, would be exactly destroyed while it was moving from syzygy to quadrature, so that no permanent alteration of areal velocity would be produced.

In reality, however, the magnitude of the disturbing force by which this inequality is caused, depends in some degree on the eccentricity of the earth's orbit, and as this is continually diminishing, the central disturbing forces at equal angular distances on opposite sides of conjunction will not be exactly equal. Hence the orbit will no longer be symmetrically situated with respect to the line of conjunction. Now the change of areal velocity produced by the tangential force at any point, depends partly on the value of the radius vector at that point, and consequently the effects of the tangential force before and after conjunction will no longer exactly balance each other.

The other inequalities of the moon's motion will be similarly modified, especially those which depend, more directly, on the eccentricity of the earth's orbit, so that each of them gives rise to an uncompensated change of the areal velocity.

Since the distortion in the form of the orbit just pointed out is due to the alteration of the disturbing force consequent upon a change in the eccentricity of the earth's orbit, and it is by virtue of this distortion that the tangential force produces a permanent change in the rate of description of areas, it follows that this alteration

of the areal velocity will be of the order of the square of the disturbing force multiplied by the rate of change of the earth's eccentricity.

It is evident that the amount of the acceleration of the moon's mean motion will be directly affected by this alteration of areal velocity.

5. Having thus briefly indicated the way in which the effect now treated of originates, I will proceed with the analytical investigation of its amount.

In the present communication, however, I shall confine my attention to the principal term of the change thus produced in the acceleration of the moon's motion, deferring to another, though I hope not a distant, opportunity, the fuller development of this subject, as well as the consideration of the secular variations of the other elements of the moon's orbit arising from the same cause.

In what follows, the notation, except when otherwise explained, is the same as that of DAMOISEAU's "Théorie de la Lune."

6. If we suppose the moon to move in the plane of the ecliptic, and also neglect the terms depending on the sun's parallax, the differential equations of the moon's motion become

$$0=\frac{d^2u}{dv^2}+u-\frac{1}{h^2}+\frac{m'u'^3}{2h^2u^3}+\frac{3}{2}\frac{m'u'^3}{h^2u^3}\cos(2v-2v')$$
$$-\frac{3}{2}\frac{m'u'^3}{h^2u^4}\frac{du}{dv}\sin(2v-2v')-\frac{3m'}{h^2}\left(u+\frac{d^2u}{dv^2}\right)\int\frac{u'^3dv}{u^4}\sin(2v-2v')$$

$$\frac{dt}{dv}=\frac{1}{hu^2}+\frac{3}{2}\frac{m'}{h^3u^3}\int\frac{u'^3dv}{u^4}\sin(2v-2v')+\frac{27}{8}\frac{m'^2}{h^5u^2}\left[\int\frac{u'^3dv}{u^4}\sin(2v-2v')\right]^2.$$

In the solution usually given of these equations, u is expressed by means of a constant part and a series involving *cosines* of angles composed of multiples of $2v-2mv$, $cv-\varpi$, and $c'mv-\varpi'$; also t is expressed by means of a part proportional to v and a series involving *sines* of the same angles; the coefficients of the periodic terms being functions of m, e and e'. Now if e' be a constant quantity, this is the true form of the solution, but if e' be variable, it is impossible to satisfy the differential equations without adding to the expression for u a series of small supplementary terms depending on the *sines* of the angles whose cosines are already involved in it, and to that for t, similar terms depending on the *cosines* of the same angles, the coefficients of these new terms involving $\frac{de'}{dt}$ as a factor.

The quantity $\int\frac{u'^3dv}{u^4}\sin(2v-2v')$, which occurs in the above equations, is proportional to the variable part of the square of the areal velocity, and consists, in the ordinary theory, of a series of periodic terms involving *cosines* of the angles above mentioned. In consequence, however, of the existence of the new terms just described, there will be added to it a series of small terms involving *sines* of the same angles, together with a non-periodic part of the form $\int He'de'$ or $\frac{1}{2}He'^2$. The introduction of this term

will evidently change the relation between the non-periodic part of $\frac{dt}{dv}$ and e'^2, upon which the secular acceleration depends.

7. We must commence by finding the new terms to be added to the ordinary expression for u.

For the sake of simplification we will neglect the eccentricity of the moon's orbit.

Let $\frac{1}{a}$ denote the non-periodic part of u, and $\frac{1}{a}+\delta u$ the complete value.

Then by substitution in the equation for u, making use of DAMOISEAU's developments of the undisturbed values of the several functions of u, u', and $v-v'$ which occur in it, putting $h^2=a_1$, and writing, for convenience, mv instead of $\int m dv+\lambda$, and $c'mv$ instead of $c'\int m dv+\lambda-w'$ (as in PLANA, vol. i. p. 322), we obtain

$$
\begin{aligned}
0=&-\frac{d^2\left(\frac{1}{a}\right)}{dv^2}+\frac{1}{a}-\frac{1}{a_1}+\frac{d^2\delta u}{dv^2}+\delta u\\
&+\frac{1}{2}\frac{\overline{m}^2}{a_1}\left(1+\frac{3}{2}e'^2\right)+\frac{3}{2}\frac{\overline{m}^2}{a_1}a'\delta u'+\frac{3}{2}\frac{\overline{m}^2}{a_1}e'\cos c'mv-\frac{3}{2}\frac{\overline{m}^2}{a_1}\{1+3e'\cos c'mv\}a\delta u\\
&-\frac{3}{2}\overline{m}^2\frac{a}{a_1}\frac{d\left(\frac{1}{a}\right)}{dv}\sin(2v-2mv)+\frac{3}{2}\frac{\overline{m}^2}{a_1}\left(1-\frac{5}{2}e'^2\right)\cos(2v-2mv)\\
&+\frac{21}{4}\frac{\overline{m}^2}{a_1}e'\cos(2v-2mv-c'mv)-\frac{3}{4}\frac{\overline{m}^2}{a_1}e'\cos(2v-2mv+c'mv)\\
&-\frac{3\overline{m}^2}{a_1}\int dv\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right.\\
&\qquad\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right\}\\
&-\frac{9}{2}\frac{\overline{m}^2}{a_1}\left\{\left(1-\frac{5}{2}e'^2\right)\cos(2v-2mv)+\frac{7}{2}e'\cos(2v-2mv-c'mv)\right.\\
&\qquad\left.-\frac{1}{2}e'\cos(2v-2mv+c'mv)\right\}a\delta u\\
&-\frac{3}{2}\frac{\overline{m}^2}{a_1}\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right.\\
&\qquad\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right\}\frac{d(a\delta u)}{dv}\\
&+12\frac{\overline{m}^2}{a_1}\int dv\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right.\\
&\qquad\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right\}a\delta u\\
&-\frac{3\overline{m}^2}{a_1}\left\{\frac{d^2(a\delta u)}{dv^2}+a\delta u\right\}\int dv\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right.\\
&\qquad\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right\}.
\end{aligned}
$$

8. Also, assume

$$a\delta u = m^2\left(1-\frac{5}{2}e'^2\right)\cos(2\nu-2m\nu)+a_{30}\frac{e'de'}{ndt}\sin(2\nu-2m\nu)$$

$$-\frac{3}{2}m^2e'\cos c'm\nu+a_{16}\frac{de'}{ndt}\sin c'm\nu$$

$$+\frac{7}{2}m^2e'\cos(2\nu-2m\nu-c'm\nu)+a_{33}\frac{de'}{ndt}\sin(2\nu-2m\nu-c'm\nu)$$

$$-\frac{1}{2}m^2e'\cos(2\nu-2m\nu+c'm\nu)+a_{34}\frac{de'}{ndt}\sin(2\nu-2m\nu+c'm\nu),$$

where the coefficients of the terms involving cosines are those given by the ordinary theory, and a_{30}, a_{16}, a_{33}, and a_{34} are numerical quantities to be determined.

9. In developing the terms of the above equation, by the substitution of this value of $a\delta u$, the quantity $\frac{de'}{dt}$ may be considered constant, and $\frac{de'}{d\nu}$ must be expressed in terms of it.

Thus $\frac{de'}{d\nu}=\frac{ndt}{d\nu}\frac{de'}{ndt}$

$$=\frac{de'}{ndt}\left\{1-\frac{11}{4}m^2\cos(2\nu-2m\nu)-\frac{77}{8}m^2e'\cos(2\nu-2m\nu-c'm\nu)\right.$$

$$\left.+\frac{11}{8}m^2e'\cos(2\nu-2m\nu+c'm\nu)\right\}.$$

Also, integrating by parts, and putting 2 instead of $2-2m$, $2-3m$, and $2-m$ in the divisors introduced by integration, since we only want to find the terms of the lowest order which are multiplied by $\frac{de'}{dt}$, we obtain

$$-\frac{3\overline{m}^2}{a_{,}}\int d\nu\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2\nu-2m\nu)+\frac{7}{2}e'\sin(2\nu-2m\nu-c'm\nu)\right.$$

$$\left.-\frac{1}{2}e'\sin(2\nu-2m\nu+c'm\nu)\right\}$$

$$=\frac{3}{2}\frac{\overline{m}^2}{a_{,}}\left(1-\frac{5}{2}e'^2\right)\cos(2\nu-2m\nu)+\frac{21}{4}\frac{\overline{m}^2}{a_{,}}e'\cos(2\nu-2m\nu-c'm\nu)$$

$$-\frac{3}{4}\frac{\overline{m}^2}{a_{,}}e'\cos(2\nu-2m\nu+c'm\nu)$$

$$+\frac{15}{2}\frac{\overline{m}^2}{a_{,}}\int d\nu\frac{e'de'}{ndt}\frac{ndt}{d\nu}\cos(2\nu-2m\nu)-\frac{21}{4}\frac{\overline{m}^2}{a_{,}}\int d\nu\frac{de'}{ndt}\frac{ndt}{d\nu}\cos(2\nu-2m\nu-c'm\nu)$$

$$+\frac{3}{4}\frac{\overline{m}^2}{a_{,}}\int d\nu\frac{de'}{ndt}\frac{ndt}{d\nu}\cos(2\nu-2m\nu+c'm\nu).$$

And $a'\delta u'=3m^2e'\sin c'm\nu[-e'\sin c'm\nu]$

$$=-\frac{3}{2}m^2e'^2,$$

retaining only the term which will be required.

10. When the proper substitutions are made, the terms involving cosines destroy each other, as in the usual theory, and by equating to zero the terms involving the sines, we obtain

$$20m^3-3a_{30}+\frac{15}{4}m^2=0,$$

or
$$3a_{30}=\frac{95}{4}m^3 \quad \therefore a_{30}=\frac{95}{12}m^2$$

$$3m^3+a_{16}=0 \quad \therefore a_{16}=-3m^3$$

$$-14m^3-3a_{33}-\frac{21}{8}m^3=0,$$

or
$$3a_{33}=-\frac{133}{8}m^3 \quad \therefore a_{33}=-\frac{133}{24}m^2$$

$$2m^3-3a_{34}+\frac{3}{8}m^3=0,$$

or
$$3a_{34}=\frac{19}{8}m^3 \quad \therefore a_{34}=\frac{19}{24}m^3.$$

11. In order to obtain the relation between a and a_1, we must substitute the value just found for $a\delta u$, in the same equation, and equate to zero the non-periodic part, observing that the terms

$$12\frac{\overline{m}^2}{a_1}\int dv\left\{(1-\frac{5}{2}e'^2)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right.$$
$$\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right\}a\delta u$$

give

$$\frac{12\overline{m}^2}{a_1}\int dv\left\{\frac{95}{24}m^3\frac{e'de'}{ndt}-\frac{931}{96}m^3\frac{e'de'}{ndt}-\frac{19}{96}m^3\frac{e'de'}{ndt}\right\}$$
$$=-\frac{285}{4}\frac{m^4}{a_1}\int ndt\frac{e'de'}{ndt}\text{ nearly,}$$
$$=-\frac{285}{8}\frac{m^4}{a_1}e'^2\text{ as their non-periodic part.}$$

Also the terms

$$\frac{15}{2}\frac{\overline{m}^2}{a_1}\int dv\frac{e'de'}{ndt}\frac{ndt}{dv}\cos(2v-2mv)-\frac{21}{4}\frac{\overline{m}^2}{a_1}\int dv\frac{de'}{ndt}\frac{ndt}{dv}\cos(2v-2mv-c'mv)$$
$$+\frac{3}{4}\frac{\overline{m}^2}{a_1}\int dv\frac{de'}{ndt}\frac{ndt}{dv}\cos(2v-2mv+c'mv)$$

of art. 9. similarly give

$$\frac{15}{2}\frac{\overline{m}^2}{a_1}\int dv\left(-\frac{11}{8}m^3\frac{e'de'}{ndt}\right)-\frac{21}{4}\frac{\overline{m}^2}{a_1}\int dv\left(-\frac{77}{16}m^3\frac{e'de'}{ndt}\right)+\frac{3}{4}\frac{\overline{m}^2}{a_1}\int dv\left(\frac{11}{16}m^3\frac{e'de'}{ndt}\right)$$
$$=-\frac{165}{32}\frac{m^4}{a_1}e'^2+\frac{1617}{128}\frac{m^4}{a_1}e'^2+\frac{33}{128}\frac{m^4}{a_1}e'^2\text{ nearly}$$
$$=\frac{495}{64}\frac{m^4}{a_1}e'^2\text{ as their non-periodic part.}$$

12. Hence we obtain

$$0=\frac{1}{a}-\frac{1}{a_1}+\frac{1}{2}\frac{\overline{m}^2}{a_1}\left(1+\frac{3}{2}e'^2\right)-\frac{9}{4}\frac{m^4}{a_1}e'^2+\frac{495}{64}\frac{m^4}{a_1}e'^2+\frac{27}{8}\frac{m^4}{a_1}e'^2$$
$$-\frac{9}{4}\frac{m^4}{a_1}(1-5e'^2)-\frac{441}{16}\frac{m^4}{a_1}e'^2-\frac{9}{16}\frac{m^4}{a_1}e'^2$$
$$+\frac{3}{2}\frac{m^4}{a_1}(1-5e'^2)+\frac{147}{8}\frac{m^4}{a_1}e'^2+\frac{3}{8}\frac{m^4}{a_1}e'^2-\frac{285}{8}\frac{m^4}{a_1}e'^2$$
$$-\frac{9}{4}\frac{m^4}{a_1}(1-5e'^2)-\frac{441}{16}\frac{m^4}{a_1}e'^2-\frac{9}{16}\frac{m^4}{a_1}e'^2,$$

or
$$0=\frac{1}{a}-\frac{1}{a_1}\left\{1-\frac{1}{2}\overline{m}^2-\frac{3}{4}\overline{m}^2e'^2+3m^4+\frac{3153}{64}m^4e'^2\right\}.$$

Now $\overline{m}^3=\frac{m^2}{(1+p)^3}$ in PLANA's notation, or (substituting the value of p given in PLANA, vol. ii. p. 855).

$$\overline{m}^3=m^3\left(1-\frac{1}{2}m^2-\frac{3}{4}m^3e'^2\right) \text{ nearly,}$$

$$\therefore \frac{1}{a}=\frac{1}{a_1}\left\{1-\frac{1}{2}m^2+\frac{13}{4}m^4-\frac{3}{4}m^3e'^2+\frac{3201}{64}m^4e'^2\right\}$$

and
$$a^2=a_1^2\left\{1+m^2-\frac{23}{4}m^4+\frac{3}{2}m^3e'^2-\frac{3129}{32}m^4e'^2\right\}.$$

13. Again, by substitution in the equation for $\frac{dt}{dv}$, we obtain

$$\frac{dt}{dv}=\frac{a^2}{\sqrt{a_1}}\Bigg\{1-2a\delta u+\frac{3}{2}m^4(1-5e'^2)+\frac{27}{8}m^4e'^2+\frac{147}{8}m^4e'^2+\frac{3}{8}m^4e'^2$$
$$+\frac{3}{2}\overline{m}^3\frac{a}{a_1}\int dv\left[\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right.$$
$$\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right]$$
$$-3\overline{m}^3\frac{a}{a_1}a\delta u\int dv\left[\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right.$$
$$\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right]$$
$$-6\overline{m}^3\frac{a}{a_1}\int dv\left[\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right.$$
$$\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right]a\delta u$$
$$+\frac{27}{8}\overline{m}^4\left(\frac{a}{a_1}\right)^2\left\{\int dv\left[\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right.\right.$$
$$\left.\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right]\right\}^2\Bigg\}.$$

14. Develope this equation as before, retaining m^4 only when it occurs in the non-periodic part, and we have

$$\frac{dt}{d\nu}=\frac{a^3}{\sqrt{a_i}}\Big\{1-2a\delta u+\frac{3}{2}m^4+\frac{3}{4}m^4(1-5e'^2)+\frac{27}{64}m^4(1-5e'^2)-\frac{495}{128}m^4e'^2$$
$$+\frac{117}{8}m^4e'^2+\frac{147}{16}m^4e'^2+\frac{3}{16}m^4e'^2+\frac{285}{16}m^4e'^2+\frac{1323}{256}m^4e'^2+\frac{27}{256}m^4e'^2$$
$$-\frac{3}{4}m^3\Big(1-\frac{5}{2}e'^2\Big)\cos(2\nu-2m\nu)-\frac{21}{8}m^3e'\cos(2\nu-2m\nu-c'm\nu)$$
$$+\frac{3}{8}m^3e'\cos(2\nu-2m\nu+c'm\nu)$$
$$-\frac{15}{8}m^3\frac{e'de'}{ndt}\sin(2\nu-2m\nu)+\frac{21}{16}m^3\frac{de'}{ndt}\sin(2\nu-2m\nu-c'm\nu)$$
$$-\frac{3}{16}m^3\frac{de'}{ndt}\sin(2\nu-2m\nu+c'm\nu)\Big\},$$

or

$$\frac{dt}{d\nu}=\frac{a^2}{\sqrt{a_i}}\Big\{1+\frac{171}{64}m^4+\frac{2391}{64}m^4e'^2$$
$$-\frac{11}{4}m^2\Big(1-\frac{5}{2}e'^2\Big)\cos(2\nu-2m\nu)-\frac{425}{24}m^3\frac{e'de'}{ndt}\sin(2\nu-2m\nu)$$
$$+3m^3e'\cos c'm\nu+6m^3\frac{de'}{ndt}\sin c'm\nu$$
$$-\frac{77}{8}m^3e'\cos(2\nu-2m\nu-c'm\nu)+\frac{595}{48}m^3\frac{de'}{ndt}\sin(2\nu-2m\nu-c'm\nu)$$
$$+\frac{11}{8}m^3e'\cos(2\nu-2m\nu+c'm\nu)-\frac{85}{48}m^3\frac{de'}{ndt}\sin(2\nu-2m\nu+c'm\nu)\Big\}.$$

15. Substitute the value before found for a^2 in terms of a_i^2;

$$\therefore\quad \frac{dt}{d\nu}=a_i^3\Big\{1+m^2-\frac{197}{64}m^4+\frac{3}{2}m^3e'^2-\frac{3867}{64}m^4e'^2$$
$$-\frac{11}{4}m^2\Big(1-\frac{5}{2}e'^2\Big)\cos(2\nu-2m\nu)-\frac{425}{24}m^3\frac{e'de'}{ndt}\sin(2\nu-2m\nu)$$
$$+3m^3e'\cos c'm\nu+6m^3\frac{de'}{ndt}\sin c'm\nu$$
$$-\frac{77}{8}m^3e'\cos(2\nu-2m\nu-c'm\nu)+\frac{595}{48}m^3\frac{de'}{ndt}\sin(2\nu-2m\nu-c'm\nu)$$
$$+\frac{11}{8}m^3e'\cos(2\nu-2m\nu+c'm\nu)-\frac{85}{48}m^3\frac{de'}{ndt}\sin(2\nu-2m\nu+c'm\nu)\Big\}.$$

16. Now, put $\quad \frac{1}{n}=a_i^3\Big\{1+m^2-\frac{197}{64}m^4+\frac{3}{2}m^3e'^2-\frac{3867}{64}m^4e'^2\Big\}$,

multiply by n, and integrate;

$$\therefore\ \int ndt=\nu-\frac{11}{8}m^2\Big(1-\frac{5}{2}e'^2\Big)\sin(2\nu-2m\nu)+\frac{295}{24}m^3\frac{e'de'}{ndt}\cos(2\nu-2m\nu)$$
$$+3me'\sin c'm\nu+3\frac{de'}{ndt}\cos c'm\nu$$
$$-\frac{77}{16}m^3e'\sin(2\nu-2m\nu-c'm\nu)-\frac{413}{48}m^3\frac{de'}{ndt}\cos(2\nu-2m\nu-c'm\nu)$$
$$+\frac{11}{16}m^3e'\sin(2\nu-2m\nu+c'm\nu)+\frac{59}{48}m^3\frac{de'}{ndt}\cos(2\nu-2m\nu+c'm\nu).$$

17. In the expression for $\frac{1}{n}$ just found, $a_{,}$ is absolutely constant, but e' is variable, consequently n will vary, and therefore m likewise, which is connected with it by the equation $m=\frac{n'}{n}$.

Taking the variation of the equation for n, and observing that $\frac{\delta m}{m}=-\frac{\delta n}{n}$, we have

$$0=\frac{\delta n}{n}(1-m^2)+\left(\frac{3}{2}m^2-\frac{3867}{64}m^4\right)\delta(e'^2),$$

$$\therefore\quad \frac{\delta n}{n}=-\left(\frac{3}{2}m^2-\frac{3771}{64}m^4\right)\delta(e'^2).$$

Therefore, if N be the initial value of n, and E′ the corresponding value of e',

$$n=\mathrm{N}-\left(\frac{3}{2}m^2-\frac{3771}{64}m^4\right)n(e'^2-\mathrm{E}'^2),$$

and
$$\int ndt=\mathrm{N}t+\varepsilon-\left(\frac{3}{2}m^2-\frac{3771}{64}m^4\right)\int(e'^2-\mathrm{E}'^2)ndt.$$

Hence the expression for the true longitude in terms of the mean, contains the secular equation

$$-\left(\frac{3}{2}m^2-\frac{3771}{64}m^4\right)\int(e'^2-\mathrm{E}'^2)ndt.$$

18. According to PLANA, the corresponding terms in the expression for the secular equation are

$$-\left(\frac{3}{2}m^2-\frac{2187}{128}m^4\right)\int(e'^2-\mathrm{E}'^2)ndt.$$

Hence we see that the terms now taken into consideration have the effect of making the second term of the secular equation more than three times as great as it would otherwise be. Of course, the succeeding terms will also be materially changed.

The principal term of the correction to be applied to PLANA's value of the secular acceleration is therefore

$$\frac{5355}{128}m^4\int(e'^2-\mathrm{E}'^2)ndt.$$

Now
$$\int(e'^2-\mathrm{E}'^2)ndt=-1270''\left(\frac{t}{100}\right)^2 \text{ nearly,}$$

where t is expressed in years; therefore the numerical value of this term is

$$-1''{\cdot}66\left(\frac{t}{100}\right)^2.$$

This result will serve to give an idea of the numerical importance of the new terms to be added to the received value of the secular acceleration, and probably will not differ widely from the complete correction; though in order to obtain a value sufficiently accurate to be definitively used in the calculation of ancient eclipses, the approximation must be carried considerably further.

The new periodic terms added to the moon's longitude are perfectly insignificant, the coefficient of that involving $\cos c'm\nu$, which is by far the largest of them, only amounting to $0''{\cdot}003$.

19. Transforming the expressions found above, so as to obtain the moon's longitude and radius vector in terms of the time, and writing for convenience nt instead of $\int ndt+\varepsilon$, mnt instead of $mnt+\varepsilon'$, and $c'mnt$ instead of $c'mnt+\varepsilon'-\varpi'$, we have

$$\begin{aligned}
v = nt &+ \frac{11}{8}m^2\left(1-\frac{5}{2}e'^2\right)\sin(2-2m)nt - \frac{74}{3}m^2\frac{e'de'}{ndt}\cos(2-2m)nt \\
&- 3me'\sin c'mnt - 3\frac{de'}{ndt}\cos c'mnt \\
&+ \frac{77}{16}m^2e'\sin(2-2m-c'm)nt + \frac{215}{48}m^2\frac{de'}{ndt}\cos(2-2m-c'm)nt \\
&- \frac{11}{16}m^2e'\sin(2-2m+c'm)nt - \frac{257}{48}m^2\frac{de'}{ndt}\cos(2-2m+c'm)nt
\end{aligned}$$

$$\begin{aligned}
\frac{a}{r} = au = 1 &- \frac{11}{8}m^4 - \frac{201}{16}m^4e'^2 \\
&+ m^2\left(1-\frac{5}{2}e'^2\right)\cos(2-2m)nt + \frac{203}{12}m^2\frac{e'de'}{ndt}\sin(2-2m)nt \\
&- \frac{3}{2}m^2e'\cos c'mnt - 3m^2\frac{de'}{ndt}\sin c'mnt \\
&+ \frac{7}{2}m^2e'\cos(2-2m-c'm)nt - \frac{61}{24}m^2\frac{de'}{ndt}\sin(2-2m-c'm)nt \\
&- \frac{1}{2}m^2e'\cos(2-2m+c'm)nt + \frac{91}{24}m^2\frac{de'}{ndt}\sin(2-2m+c'm)nt.
\end{aligned}$$

20. The existence of the new terms in the expressions for the moon's coordinates occurred to me some time since, when I was engaged in thinking over a new method of treating the lunar theory, though I did not then perceive their important bearing on the value of the secular equation.

My attention was first directed to this latter subject while endeavouring to supply an omission in the theory of the moon given by PONTÉCOULANT in his "Théorie Analytique." In this valuable work, the author, following the example originally set by Sir J. LUBBOCK in his Tracts on the Lunar Theory, obtains directly the expressions for the moon's coordinates in terms of the time, which are found in PLANA's theory by means of the reversion of series. With respect to the secular acceleration of the mean motion, however, PONTÉCOULANT unfortunately adopts PLANA's result without examination. On performing the calculation requisite to complete this part of the theory, I was surprised to find that the second term of the expression for the secular acceleration thus obtained, not only differed totally in magnitude from the corresponding term given by PLANA, but was even of a contrary sign. My previous researches, however, immediately led me to suspect what was the origin of this discordance, and when both processes were corrected by taking into account the new terms whose existence I had already recognized, I had the satisfaction of finding a perfect agreement between the results.

June 16th, 1853.

XVIII. *On a Theory of the Syzygetic* relations of two rational integral functions, comprising an application to the Theory of* STURM'S *Functions, and that of the greatest Algebraical Common Measure.*

By J. J. SYLVESTER, *M.A. Dub., F.R.S., Barrister at Law, and formerly Professor of Natural Philosophy in University College, London.*

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INTRODUCTION.

"How charming is divine philosophy!
Not harsh and crabbed as dull fools suppose,
But musical as is Apollo's lute,
And a perpetual feast of nectar'd sweets,
Where no crude surfeit reigns!"—COMUS.

IN the first section of the ensuing memoir, which is divided into five sections, I consider the nature and properties of the residues which result from the ordinary process of successive division (such as is employed for the purpose of finding the greatest common measure) applied to $f(x)$ and $\varphi(x)$, two perfectly independent rational integral functions of x. Every such residue, as will be evident from considering the mode in which it arises, is a syzygetic function of the two given functions; that is to say, each of the given functions being multiplied by an appropriate other function of a given degree in x, the sum of the two products will express a corresponding residue. These multipliers, in fact, are the numerators and denominators to the successive convergents to $\frac{\varphi x}{fx}$ expressed under the form of a continued fraction. If now we proceed *à priori* by means of the given conditions as to the degree in (x) of the multipliers and of any residue, to determine such residue, we find, as shown in art. (2.), that there are as many homogeneous equations to be solved as there are constants to be determined; accordingly, with the exception of one arbitrary factor which enters into the solution, the problem is definite; and if it be further agreed that the quantities entering into the solution shall be of the lowest possible dimensions in respect of the coefficients of f and φ, and also of the lowest numerical denomination, then the problem (save as to the algebraical sign of *plus* or *minus*) becomes absolutely determinate, and we can assign the numbers of the dimensions for the respective residues and syzygetic multipliers. The residues given by the method of successive division are easily seen not

* *Conjugate* would imply something very different from *Syzygetic*, viz. a theory of the Invariantive properties of a system of two algebraical functions.

to be of these lowest dimensions; accordingly there must enter into each of them a certain unnecessary factor, which, however, as it cannot be properly called irrelevant, I distinguish by the name of the Allotrious Factor. The successive residues, when divested of these allotrious factors, I term the Simplified Residues, and in article (3.) and (4.) I express the allotrious factors of each residue in terms of the leading coefficients of the preceding simplified residues of f and φ. In article (5.) I proceed to determine by a direct method these simplified residues in terms of the coefficients of f and φ. Beginning with the case where f and φ are of the same dimensions (m) in x, I observe that we may deduce from f and φ m linearly independent functions of x each of the degree $(m-1)$ in x, all of them syzygetic functions of f and φ (vanishing when these two simultaneously vanish), and with coefficients which are made up of terms, each of which is the product of one coefficient of f and one coefficient of φ. These, in fact, are the very same (m) functions as are employed in the method which goes by the name of Bezout's abridged method to obtain the resultant to (*i. e.* the result of the elimination of x performed upon) f and φ. As these derived functions are of frequent occurrence, I find it necessary to give them a name, and I term them the (m) Bezoutics or Bezoutian Primaries; from these (m) primaries m Bezoutian secondaries may be deduced by eliminating linearly between them in the order in which they are generated,—first, the highest power of x between two, then the two highest powers of x between three, and finally, all the powers of x between them all: along with the system thus formed it is necessary to include the first Bezoutian primary, and to consider it accordingly as being also the first Bezoutian secondary; the last Bezoutian secondary is a constant identical with the Resultant of f and φ. When the m times m coefficients of the Bezoutian primaries are conceived as separated from the powers of x and arranged in a square, I term such square the Bezoutic square. This square, as shown in art. (7.), is symmetrical above one of its diagonals, and corresponds therefore (as every symmetrical matrix must do) to a homogeneous quadratic function of (m) variables of which it expresses the determinant. This quadratic function, which plays a great part in the last section and in the theory of real roots, I term the Bezoutiant; it may be regarded as a species of generating function. Returning to the Bezoutic system, I prove that the Bezoutian secondaries are identical in form with the successive simplified residues. In art. (6.) I extend these results to the case of f and φ being of different dimensions in x. In art. (7.) I give a mechanical rule for the construction of the Bezoutic square. In art. (8.) I show how the theory of $f(x)$ and $\varphi(x)$, where the latter is of an inferior degree to f, may be brought under the operation of the rule applicable to two functions of the same degree at the expense of the introduction of a known and very simple factor, which in fact will be a constant power of the leading coefficient in $f(x)$. In art. (9.) I give another method of obtaining directly the simplified residues in all cases. In art. (10.) I present the process of successive division under its most general aspect. In arts. (11.) and (12.) I demonstrate the identity of the *algebraical sign* of the Bezoutian secondaries with

that of the simplified residues, generated by a process corresponding to the development of $\frac{\varphi x}{fx}$ under the form of an *improper* continued fraction (where the negative sign takes the place of the positive sign which connects the several terms of an ordinary continual function). As the simplified residue is obtained by driving out an allotrious factor, the signs of the former will of course be governed by the signs accorded by previous convention to the latter; the convention made is, that the allotrious factors shall be taken with a sign which renders them always *essentially positive* when the coefficients of the given functions are real. I close the section with remarking the relation of the syzygetic factors and the residues to the convergents of the continued fraction which expresses $\frac{\varphi x}{fx}$, and of the continued fraction which is formed by reversing the order of the quotients in the first named fraction.

In the second section I proceed to express the residues and syzygetic multipliers in terms of the roots and factors of the given functions; the method becoming as it may be said *endoscopic* instead of being *exoscopic**, as in the first section. I begin in arts. (14.) and (15.) with obtaining in this way, under the form of a sum or double sum of terms involving factors and roots of f and φ, and certain arbitrary functions of the roots in each term, a general representative, or to speak more precisely, a group of general representatives for a conjunctive of any given degree in x to f and φ, *i. e.* a rational integral function of x, which is the sum of the products of f and φ multiplied respectively by rational integral functions of x, so as to vanish of necessity when f and φ simultaneously vanish. This variety of representatives refers not merely to the appearance of arbitrary functions, but to an essential and precedent difference of representation quite irrespective of such arbitrariness.

In articles (16.), (17.), (18.), (19.), (20.), (21.), I show how the arbitrary form of function entering into the several terms of any one (at pleasure) of the formulæ that represent a conjunctive of any given degree may be assigned, so as to make such conjunctive identical in form with a simplified residue of the same degree. The form of arbitrary function so assigned, it may be noticed, is a fractional function of the roots, so that the expression becomes a sum or double sum of fractions. I first prove in arts. (16.), (17.) that such sum is essentially integral, and I determine the *weight* of its leading coefficient in respect of the roots of f and φ (this weight being measured

* These words admit of an extensive and important application in analysis. Thus the methods for resolving an equation (or to speak more accurately, for making one equation depend upon another of a simpler form) furnished by Tschirnhausen and Mr. Jerrard (although not so presented by the latter) are essentially exoscopic; on the other hand, the methods of Lagrange and Abel for effecting similar objects are endoscopic. So again, the memoir of Jacobi, "De Eliminatione," hereinafter referred to, takes the exoscopic, and the valuable "Nota ad Eliminationem pertinens" of Professor Richelot in Crelle's Journal, the endoscopic view of the subject. In the present memoir (in which the two trains of thought arising out of these distinct views are brought into mutual relation) the subject is treated (chiefly but not exclusively) under its endoscopic aspect in the second, third and fourth sections, and exoscopically in the first and last sections.

by the number of roots of f and φ conjointly, which appear in any term of such coefficient). Now in the succeeding articles I revert to the Bezoutic system of the first section, and beginning with the supposition of m and n being equal, I demonstrate that the most general form of a conjunctive of any degree in x will be a linear function of the Bezoutics, from which it is easy to deduce that the simplified residues of any given degree in x are the conjunctives whose weight in respect of the roots is a *minimum*; so that all conjunctives having that weight must be identical (to a numerical factor *près*), and any integral form of less weight apparently representing a conjunctive must be nugatory, every term vanishing identically. These results are then extended to the case of two functions of unlike degrees. The conclusion is, that the weight of the forms assumed in (16.) and (17.) being equal to the minimum weight, they must (unless they were to vanish, which is easily disproved) represent the simplified residues, or which is the same thing, the Bezoutian secondaries.

We thus obtain for each simplified residue a number of essentially distinct forms of representation, but all of which must be identical to a numerical factor *près*, a result which leads to remarkable algebraical theorems.

The number of these different formulæ depends upon the degree of the residue; there being only one for the last or constant residue, two for the last but one, three for the last but two, and so on. The formulæ continue to have a meaning when their degree in x exceeds that of f or φ; but then, as although always representing conjunctives, they no longer represent residues, this identity no longer continues to subsist. In articles (22.), (23.), (24.), (25.), I enter into some developments connected with the general formulæ in question: these, it may be observed, are all expressed by means of fractions containing in the numerator and denominator products of differences; the differences in the numerator products being taken between groups of roots of f and groups of roots of φ; and in the denominator between roots of f *inter se* and roots of φ *inter se*. A great enlargement is thus opened out to the ordinary theory of partial fractions.

In art. (26.) I find the numerical ratios between the different formulæ which represent (to a numerical factor *près*) the same simplified residue, and in arts. (27.) and (28.) I determine the relations of algebraical sign of these formulæ to the simplified residues or Bezoutian secondaries. In art. (29.) I determine the syzygetic multipliers corresponding to any given residue in terms of the factors and roots of the given functions; but the expressions for these, which are closely analogous to those for the residues, cease to be polymorphic. They are obtained separately from the syzygetic equation, and it is worthy of notice, that to obtain the one we use the first of the polymorphic expressions for the residue, and to obtain the other the opposite extremity of the polymorphic scale. In the subsequent articles of this section, by aid of certain general properties of continued fractions, I establish a theorem of reciprocity between the series of residues and either series of syzygetic multipliers.

Section III. is devoted to a determination of the values of the preceding formulæ for the residues and multipliers in the case applicable to M. STURM's theorem, where $\phi(x)$ becomes the differential derivative of fx. It becomes of importance to express the formulæ for this case in terms of their roots and factors of fx alone, without the use of the roots and factors of $f'x$, which will of course be functions of the former.

By selecting a proper form out of the polymorphic scale, the fractional terms of the series for each residue in this case become separately integral, and we obtain my well-known formulæ for the simplified residues (STURM's reduced auxiliary functions) in terms of the factors and the squared differences of partial groups of roots. This is shown in art. (35.). In art. (36.) the multiplier of $f'x$ in the syzygetic equation is expressed by formulæ of equal simplicity, and in a certain sense complementary to the former. This method, however, does not apply to obtaining expressions for the multiplier of fx in the same equation in terms of the roots and factors of fx; for the separate fractions whose sum represents any one of these factors it will be found do not admit of being expressed as integral functions of the roots and factors. To obviate this difficulty I look to the syzygetic equation itself, which contains five quantities, viz. the given function, its first differential derivative, the residue of a given degree, and the two multipliers, all of which, except the multiplier of fx, are known, or have been previously determined as rational integral functions of the roots and factors of fx. I use this equation itself for determining the fifth quantity, the multiplier in question. To perform the general operations by a direct method required for this would be impossible; the difficulty is got over by finding, by means of the syzygetic equation, the particular form that the result must assume when certain relations of equality spring up between the roots of fx; and then, by aid of these particular determinations, the general form is demonstratively inferred.

This investigation extends over arts. (38.), (39.), (40.), (41.), (42.), (43.). It turns out that the expressions for the multipliers of fx are of much greater complexity than for the multipliers of $f'x$ or for the residues. Any such multiplier consists of a sum of parts, each of which, as in the case of the residues and of the factors of $f'x$, is affected with a factor consisting of the squared differences of a group of roots; but the other factor, instead of being simply (as for the residues and factors before mentioned) a product of certain factors of fx, consists of the sum of a series of products of sums of powers by products of combinations of factors of fx, each of which series is affected with the curious anomaly of its last term, becoming augmented in a certain numerical ratio beyond what it should be, in order to be conformable to the regular flow of the preceding terms in the series*.

The fourth section opens with the establishment of two propositions concerning

* The syzygetic multipliers are identical with the numerators and denominators (expressed in their simplest form) of the successive convergents to the continued fraction which expresses $\frac{f'x}{fx}$.

quadratic functions which are made use of in the sequel. Art. (28.) contains the proof of a law which, although of extreme simplicity, I do not remember to have seen, and with which I have not found that analysts are familiar: I mean the law of the constancy of signs (as regards the number of positive and negative signs) in any sum of positive and negative squares into which a given quadratic function admits of being transformed by substituting for the variables linear functions of the variables with real coefficients. This constant *number* of positive signs which attaches to a quadratic function under all its transformations, and which is a transcendental function of the coefficients invariable for *real* substitutions, may be termed conveniently its *inertia*, until a better word be found. This inertia it is shown in art. (26.), by aid of a theorem identical with one formerly given by M. Cauchy, is measured by the number of combinations of sign in the series of determinants of which the first is the complete determinant of the function; the second, the determinant when one variable is made zero; the next, the determinant when another variable as well as the first is made zero, and so on, until all the variables are exhausted, and the determinant becomes positive unity. In art. (46.) I give some curious and interesting expressions for the residues and syzygetic multipliers, under the form of determinants communicated to me by M. Hermite; and in art. (47.) I show how, by aid of the generating function which M. Hermite employs, and of the law of inertia stated at the opening of the section, an *instantaneous* demonstration may be given of the applicability of my formulæ for M. Sturm's functions for discovering the number of real roots of fx, without any reference to the rule of common measure; and moreover, that these formulæ may be indefinitely varied, and give the generating function, out of which they may be evolved in its most general form. Had the law of inertia been familiar to mathematicians, this constructive and instantaneous method of finding formulæ for determining the number of real roots within prescribed limits would, in all probability, have been discovered long ago, as an obvious consequence of such law. I then proceed in arts. (48.) and (49.), to inquire as to the nature of the indications afforded by the successive simplified residues to two general functions f and ϕ; and I find that the succession of signs of these residues serves to determine the number of roots of f or ϕ, comprised between given limits after all pairs of roots of either function, contained within the given limits, not separated by roots of the other function, have been removed, and the operation, if necessary, repeated *toties quoties* until no two roots of either function are left unseparated by roots of the other; or in other words, until every root finally retained in one function is followed by a root of the other, or else by one of the assigned limits. The system of roots comprised between given limits thus reduced I call the effective scale of intercalations; such a scale may begin with a root of the numerator or of the denominator of $\frac{\phi x}{fx}$; and upon this and the relative magnitudes of the greatest root of ϕx and fx it will depend whether in the series of residues (among which fx and ϕx are for this purpose to be counted)

changes will be lost or gained as x passes from positive infinity to negative infinity. In art. (50.) I observe that the theory of real roots of a single function given by M. STURM's theorem is a corollary to this theory of the intercalations of real roots of two functions, depending upon the well-known law, that odd groups of the limiting function $f'x$ lie between every two consecutive real roots of fx. In art. (51.) I verify the law of reciprocity, already stated to exist between the residues of fx and φx, by an *à posteriori* method founded on the theory of intercalations. In arts. (52.), (53.), (54.), I obtain a remarkable rule, founded upon the process of common measure, for finding a superior and inferior limit in an infinite variety of ways to the roots of any given function. This method stands in a singular relation of contrast to those previously known. All previous methods (including those derived through NEWTON's Rule) proceed upon the idea of treating the function whose roots are to be limited as made up of the *sum* of parts, each of which retains a constant sign for all values of the variable external to the quantities which are to be shown to limit the roots. My method, on the other hand, proceeds upon the idea of treating the function as the product of *factors* retaining a constant sign for such values of the variable. In art. (55.), the concluding article of the fourth section, I point out a conceivable mode in which the theory of intercalations may be extended to systems of three or more functions.

In Section V. arts. (56.), (57.), I show how the *total* number of effective intercalations between the roots of two functions of the same degree is given by the *inertia* of that quadratic form which we agreed to term the Bezoutiant to f and φ; and in the following article (58.) the result is extended to embrace the case contemplated in M. STURM's theorem; that is to say, I show, that on replacing the function of x by a homogeneous function of x and y, the Bezoutiant to the two functions, which are respectively the differential derivatives of f with respect to x and with respect to y, will serve to determine by its form or *inertia* the total number of *real* roots and of *equal* roots in $f(x)$. The subject is pursued in the following arts. (59.), (60.). The concluding portion of this section is devoted to a consideration of the properties of the Bezoutiant under a purely morphological point of view; for this purpose f and φ are treated as homogeneous functions of two variables x, y, instead of being regarded as functions of x alone. In arts. (61.), (62.), (63.), it is proved that the Bezoutiant is an invariantive function of the functions from which it is derived; and in art. (64.) the important remark is added, that it is an invariant of that particular class to which I have given the name of Combinants, which have the property of remaining unaltered, not only for linear transformations of the variables, but also for linear combinations of the functions containing the variables, possessing thus a character of double invariability. In arts. (65.), (66.), I consider the relation of the Bezoutiant to the differential determinant, so called by JACOBI, but which for greater brevity I call the Jacobian. On proper substitutions being made in the Bezoutiant of the (m) variables which it contains (m being the degree in x, y of f and φ), the Bezoutiant becomes identical with the Jacobian to f and φ; but as it is afterwards

shown, this is not a property peculiar to the Bezoutiant; in fact there exists a whole family of quadratic forms of m variables, lineo-linear (like the Bezoutiant) in respect of the coefficients in f and ϕ, all of which enjoy the same property. The number of individuals of such family must evidently be infinite, because any linear combination of any two of them must possess a similar property; I have discovered, however, that the number of independent forms of this kind is limited, being equal to the number of odd integers not greater than the degree of the two functions f and ϕ. In arts. (67.) and (68.), I give the means of constructing the scale of forms, which I term the constituent or fundamental scale, of which all others of the kind are merely numerico-linear combinations. This scale does not directly include the Bezoutiant within it, and it becomes an object of interest to determine the numbers which connect the Bezoutiant with the fundamental forms; this calculation I have carried on (in arts. (69.), (70.), (71.)) from $m=1$ to $m=6$ inclusive, and added an easy method of continuing indefinitely. In this method the numbers in the linear equation corresponding to any value of m are determined successively, and each made subject to a verification before the next is determined, there being always pairs of equations which ought to bring out the same result for each coefficient.

In the next and concluding art. (72.), I remark upon the different directions in which a generalization may be sought of the subject-matter of the ideas involved in M. STURM's theorem, and of which the most promising is, in my opinion, that which leads through the theory of intercalations. Some of the theorems given by me in this paper have been enunciated by me many years ago, but the demonstrations have not been published, nor have they ever before been put together and embodied in that compact and organic order in which they are arranged in this memoir,—the fruit of much thought and patient toil, which I have now the honour of presenting to the Royal Society.

June 16, 1853.

In a supplemental part to the third section I have given expressions in terms of the roots of ϕx *and* fx *for the* quotients *which arise in developing* $\frac{\phi x}{fx}$ *under the form of a continued fraction, and some remarkable properties concerning these* quotients. *In a supplemental part to the fourth section I have given an extended theory of my new method of finding limits to the real roots of any algebraical equation. This method, so extended, possesses a marked feature of distinction from all preceding methods used for the same purpose, inasmuch as it admits in every case of the limits being brought up into actual coincidence with the extreme roots, whereas in other methods a wide and arbitrary interval is in general necessarily left between the roots and the limits.*

Section I.

On the complete and simplified residues generated in the process of developing under the form of a continued fraction, an ordinary rational algebraical fraction.

Art. (1.). Let P and Q be two rational integral functions of x, and suppose that the process of continued successive division leads to the equations

$$\left.\begin{aligned} P &= M_0 Q + R_1 = 0 \\ Q &= M_1 R_1 + R_2 = 0 \\ R_1 &= M_2 R_2 + R_3 = 0 \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1.)$$

so that

$$\frac{Q}{P} = \frac{1}{M_0 -}\,\frac{1}{M_1 -}\,\frac{1}{M_2 -}\,\&c. \quad \ldots\ldots\ldots\ldots \quad (2.)$$

which is what I propose to call an improper continued fraction, differing from a proper only in the circumstance of the successive terms being connected by negative instead of positive signs.

M_0, M_1, M_2, &c., R_1, R_2, R_3, &c. are, of course, functions of x: the latter we may agree to call the 1st, 2nd, 3rd, &c. residues (in order to avoid the use of the longer term "residues with the signs changed"); and by way of distinction from what they become when certain factors are rejected, we may call R_1, R_2, R_3, &c. the complete residues. Each such complete residue will in general be of the form $\frac{N_i \cdot \varrho_i}{D_i}$, N_i and D_i being integral functions of the coefficients only of P and Q, but ϱ_i an integral function of these coefficients, and of x: ρ_i may then be termed the ith simplified residue, and $\frac{N_i}{D_i}$ the ith *allotrious* factor. Suppose P to be of m and Q of n dimensions in x, and $m-n=e$, the process of continued division may be so conducted, that all the residues may contain only integer powers of x; and we may upon this supposition make M_0 of e dimensions, and M_1, M_2, M_3, &c. each of one dimension only in x; so that R_1, R_2, R_3, ... will be respectively of $(n-1)$, $(n-2)$, $(n-3)$, &c. dimensions in x.

P and Q are supposed to be perfectly unrelated, and each the most general function that can be formed of the same degree. From (1.) we obtain

$$\left.\begin{aligned} R_1 &= M_0 . Q - P \\ R_2 &= M_1 R_1 - Q \\ &= (M_0 M_1 - 1) Q - M_1 . P \\ R_3 &= (M_0 M_1 M_2 + M_0 + M_2) Q - (M_1 M_2 - 1) P \\ \&c. &= \&c. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3.)$$

and in general we shall have

$$R_\iota = Q_\iota . Q + P_\iota . P, \quad \ldots \ldots \ldots \ldots \ldots \quad (4.)$$

where it is evident that Q_ι will be of $e+(\iota-1)$, and P_ι of $(\iota-1)$ dimensions in x.

Art. (2.) Hence it follows that the ratios $P_\iota : Q_\iota : R_\iota$ may be ascertained by the direct application of the method of indeterminate coefficients, for Q_ι will contain $e+\iota$, and P_ι will contain ι disposable constants, making $e+2\iota$ disposable constants in all. Again, $Q_\iota . Q$ and $P_\iota P$ will each rise to the degree $n+e+2-1$ in x; but their sum R_ι is to be only of $n-\iota$ dimensions in x. Hence we have to make $(n+e+\iota-1)-(n-\iota)$, *i. e.* $e+2\iota-1$ quantities (which are linear in respect to the given coefficients in P and Q, as well as in respect to the new disposable constants in P_ι and Q_ι) all vanish, that is to say, there will be $e+2\iota-1$ linear homogeneous equations to be satisfied by means of $e+2\iota$ disposable quantities; the ratios of these latter are, therefore, *determinate*, so that we may write

$$\left.\begin{aligned} P_\iota &= \lambda_\iota . (P_\iota) \\ Q_\iota &= \lambda_\iota (Q_\iota) \\ R_\iota &= \lambda_\iota (R_\iota) \end{aligned}\right\}; \quad \ldots \ldots \ldots \ldots \ldots \quad (5.)$$

and when (P_ι), (Q_ι), (R_ι) are taken prime to one another, it is obvious that (R_ι) will be in all of $e+2\iota$ dimensions in the given coefficients, *i. e.* of ι in respect of the coefficients of P, and of $e+\iota$ in respect of those of Q: λ_ι will correspond to what I have previously called the allotrious factor; being in fact foreign to the value of R_ι as determined by means of the equation (4.), and arising only from the particular method employed to obtain it through the medium of the system (1.): it becomes a matter of some interest and importance to determine the values of this allotrious factor for different values of ι*.

* These are identical with what I termed quotients of succession in the London and Edinburgh Philosophical Magazine (December, 1839); but by an easily explicable error of inadvertence, the quantities "Q_1," "Q_2," &c. therein set out are not as they are therein stated to be, the quotients of succession or allotrious factors themselves, but the ratios of each such to the one preceding, if in the series; so that—

"Q_1" is λ_1

"Q_2" is $\frac{\lambda_2}{\lambda_1}$

"Q_3" is $\frac{\lambda_3}{\lambda_2}$

&c. . . .

This error is corrected by my distinguished friend M. Sturm (Liouville's Journal, tom. viii. 1842. Sur un théorème d'Algèbre de M. Sylvester), who appears, however, to have overlooked that I was obviously well acquainted with the existence and nature of these factors, and their essential character, of being perfect squares in the case contemplated in his memoir and my own. MM. Borchardt, Terquem, and other writers, in quoting my formulæ for M. Sturm's auxiliary functions, have thus been led into the error of alluding to them as completed by M. Sturm.

Art. (3.). This may be done by the following method, which is extremely simple, and would admit of a considerable extension in its applications, were it not beside my immediate purpose to digress from the objects set out in the title to the memoir, by entering upon an investigation of the special or singular cases which may arise in the process of forming the continued fraction, when one or more of the leading coefficients in any of the residues vanish; such an inquiry would require a more general character to be imparted to the values of the quotients and residues than I shall for my present purposes care to suppose.

Let us begin with supposing $e=1$, and write

$$\left.\begin{aligned} f&=ax^n+bx^{n-1}+cx^{n-2}+\&c.\\ \varphi&=\alpha x^{n-1}+\beta x^{n-2}+\gamma x^{n-3}+\&c.\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6.)$$

Let ψ be the first residue of $\frac{f}{\varphi}$, and ω of $\frac{\varphi}{\psi}$, and therefore of $\frac{\varphi}{\alpha^2\psi}$, so that ω is the second residue of $\frac{f}{\varphi}$.

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Let $\omega=\lambda(\omega)$, ω being entirely integer, and λ a function of the coefficients in f and φ. If we make $\lambda=\frac{N}{D}$, N and D being integer functions, D will evidently be L^2; where L denotes the first coefficient in the simplified residue $\alpha^2\varphi$, and is evidently of two dimensions in α, β, &c., and of one in a, b, &c.; $D\omega$ is therefore of $2\times2+1$, *i. e.* five dimensions in α, β, &c., and of two dimensions in a, b, &c.; but ω (by virtue of what has been observed of the equations in system (5.)) is of three dimensions in α, β, &c., and of two in a, b, &c. Hence N is of two dimensions in α, β, &c., and of none in a, b, &c. This enables us at once to perceive that $N=\alpha^2$,

$$\left.\begin{aligned} &\text{for } \psi \text{ is of the form } f-(px+q)\varphi,\\ &\text{and } \omega \text{ is of the form } \varphi-(p'x+q')\psi\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (7.)$$

But $N=0$ makes ω vanish, and therefore, upon this supposition, f and φ would appear to have a common algebraical factor ψ, that is to say, N vanishing, would appear to imply that the resultant of f and φ must vanish, so that N would appear to be contained as a factor in this general resultant, which latter is, however, clearly indecomposable into factors—a seeming paradox—the solution of which must be sought for in the fact, that the equation $N=0$ is *incompatible* with the existence of the usual equations (7.) connecting f, φ, ψ and ω: but this failure of the existence of the equations (7.) (bearing in mind that N has been shown to be a function only of the set of coefficients α, β, &c.), can only happen by reason of α vanishing whenever N vanishes; α must therefore be a root of N, or which is the same thing, N a power of (α) and hence $N=\alpha^2$.

The same result may be obtained *à posteriori* by actually performing the successive divisions; if the coefficients of any dividend be a, b, c, d, &c., and of the divisor

α, β, γ, δ, &c., the first remainder forming the second divisor will be easily seen to have for its coefficients—

$$\frac{1}{\alpha^2}\cdot\begin{vmatrix} a & b & c \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{vmatrix}, \quad \frac{1}{\alpha^3}\cdot\begin{vmatrix} a & b & d \\ 0 & \alpha & \beta \\ \alpha & \beta & \delta \end{vmatrix}, \quad \frac{1}{\alpha^3}\cdot\begin{vmatrix} a & b & e \\ 0 & \alpha & \beta \\ \alpha & \beta & \varepsilon \end{vmatrix} \text{ &c.}$$

Hence the coefficients in the next remainder (making $\begin{vmatrix} a & b & c \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{vmatrix} = m$) will be each of the form of the compound determinant,—

$$\frac{1}{m^2}\cdot\left\{\begin{matrix} \alpha & & & \beta & & & \gamma & & \\ & & & a & b & c & a & b & d \\ 0 & & & 0 & \alpha & \beta & 0 & \alpha & \gamma \\ & & & \alpha & \beta & \gamma & \alpha & \beta & \delta \\ a & b & c & a & b & d & a & b & e \\ 0 & \alpha & \beta & 0 & \alpha & \gamma & 0 & \alpha & \delta \\ \alpha & \beta & \gamma & \alpha & \beta & \delta & \alpha & \beta & \varepsilon \end{matrix}\right\}.$$

The compound determinant above written will be the first coefficient in the remainder under consideration; the subsequent coefficients will be represented by writing f, φ; g, γ, &c., respectively in lieu of e, ε. Omitting the common multiplier $\frac{1}{m^2}$, the determinant above written is equal to

$$\alpha\left\{\begin{matrix} a & b & e \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{matrix}\times\begin{matrix} a & b & e \\ 0 & \alpha & d \\ \alpha & \beta & \varepsilon \end{matrix} - \begin{matrix} a & b & d \\ 0 & \alpha & \gamma \\ \alpha & \beta & \delta \end{matrix}\times\begin{matrix} a & b & d \\ 0 & \alpha & \gamma \\ \alpha & \beta & \delta \end{matrix}\right\}$$
$$+\begin{matrix} a & b & c \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{matrix}\times\left\{\begin{matrix} a & b & d \\ \beta.0 & \alpha & \gamma \\ \alpha & \beta & \delta \end{matrix} - \gamma.\begin{matrix} a & b & c \\ 0 & \alpha & \beta \\ \alpha & \beta & \gamma \end{matrix}\right\}.$$

The last written pair of terms are together equal to

$$\begin{matrix} a & b & c \\ 0 & \alpha & \beta \\ a & \beta & \gamma \end{matrix}\times\left\{-d\beta\alpha^3+c\gamma\alpha^3+a\alpha(\beta\delta-\gamma^2)\right\},$$

which is of the form $\alpha^3 A-\alpha^2\beta^2(\beta\delta-\gamma^2)\alpha$, and the sum of the first written pair is of the form $\alpha^2 B+(a\beta^2.a\beta\delta-a\gamma\beta.a\gamma\beta)\alpha$. Hence the entire determinant is of the form $\alpha^2(A+B)$, showing that α^2 will enter as a factor into this and every subsequent coefficient in the second remainder, as previously demonstrated above.

It may, moreover, be noticed, that this remainder, when α^2 has been expelled, will for general values of the coefficients be numerically as well as literally in its lowest

terms, as evinced by the fact that there exist terms (*ex. gr.* $\alpha\alpha^2\gamma_1$) having ± 1 for their numerical part. The same explicit method might be applied to show, that if the first divisor were e degrees instead of being only one degree in x lower than the first dividend, α^{e+1} would be contained in every term of the second residue; the difficulty, however, of the proof by this method augments with the value of e; but the same result springs as an immediate consequence from the method first given, which remains good *mutatis mutandis* for the general case, as may easily be verified by the reader. Applying now this result to the functions P and Q, supposed to be of the respective degrees n and $n-e$ in x, and calling the coefficients of the leading terms in the successive simplified residues α_1, α_2, α_3, &c., and the leading coefficient in Q α, and before denoting the successive allotrious factors by λ_1, λ_2, &c., it will readily be seen that

$$\lambda_1=\frac{1}{\alpha^{e+1}} \quad \lambda_2.\lambda_1=\frac{1}{\alpha_1^2} \quad \lambda_3.\lambda_2=\frac{1}{\alpha_2^2} \quad \lambda_4.\lambda_3=\frac{1}{\alpha_3^2}, \text{ &c.,}$$

$$\textit{i. e. } \lambda_1=\frac{1}{\alpha^{e+1}} \quad \lambda_2=\frac{\alpha^{e+1}}{\alpha_1^2} \quad \lambda_3=\frac{\alpha_1^2}{\alpha^{e+1}\alpha_2^2} \quad \lambda_4=\frac{\alpha^{e+1}.\alpha_2^2}{\alpha_1^2.\alpha_3^2},$$

and in general

$$\left.\begin{aligned}\lambda_{2m+1}&=\frac{1}{\alpha^{e+1}}.\frac{\alpha_1^2.\alpha_3^2....\alpha_{2m-1}^2}{\alpha_2^2.\alpha_4^2....\alpha_{2m}^2}\\ \lambda_{2m}&=\alpha^{e+1}.\frac{1.\alpha_2^2.\alpha_4^2...\alpha_{2m-2}^2}{\alpha_1^2.\alpha_3^2.\alpha_5^2...\alpha_{2m-1}^2}\end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

Art. (4.). Strictly speaking, we have not yet fully demonstrated that the complete allotrious factors are represented by the values above given for λ, but only that these latter are contained as factors in the allotrious factors; we must further prove that there exist no other such factors. This may be shown as follows: it is obvious from the nature of the process that the complete residues will always remain of one dimension in respect of the given coefficients, *i. e.* first of one dimension in the set a, b, c, &c., and of zero dimensions in α, β, γ, &c.; then conversely, of one dimension in α, β, γ, &c., and of zero dimensions in a, b, c, &c., and so on, the residues being evidently required to conform in their dimensions to those of the first dividend and the first divisor alternately. These coefficients then are always of unit dimensions in respect to the given coefficients; whereas it has been shown (art. 2.) that the simplified residues in respect to these coefficients are successively of the dimensions $2+e$, $4+e$, $6+e$, &c.

Let the complete residue corresponding to λ_{2m} be $\mathrm{M}.\lambda_{2m}.\alpha_{2m}$,

$$\textit{i. e. } \mathrm{M}.\frac{\alpha^{e+1}}{\alpha_1^2}.\frac{\alpha_2^2}{\alpha_3^2}.\frac{\alpha_4^2}{\alpha_5^2}.....\frac{\alpha_{2m-2}^2}{\alpha_{2m-1}}.\alpha_{2m},$$

or say $\mathrm{M}.\mathrm{L}$; in passing from α_{2q} to α_{2q+1} the dimensions rise 2 units for all values of q except zero, and when $q=0$ the dimensions increase *per saltum* from 1 to $2+e$; hence the total dimensions of L in the joint coefficients will be

$$\big((e+1)-(2e+2)\big)-(m-1)4+4m+e=1,$$

and therefore M is of zero dimensions, and λ_{2m} is the complete allotrious factor. In like manner if the complete residue corresponding to λ_{2m+1} be $M.\lambda_{2m+1}.\alpha_{2m+1}$,

$$i.\,e.\ M\frac{1}{\alpha^{e+1}}\cdot\frac{\alpha_1^2}{\alpha_2^2}\cdot\frac{\alpha_3^2}{\alpha_4^2}\cdot\cdots\cdot\frac{\alpha_{2m-1}^2}{\alpha_{2m}^2}\cdot\alpha_{2m+1},$$

or say M.L, the dimensions of L will be

$$-(e+1)-m.4+\big(e+2.(2m+1)\big),\ i.\,e.\ 1,$$

and hence, as in the preceding case, M is of zero dimensions, and λ_{2m+1} is the complete allotrious factor.

Art. (5.). I proceed to show how the simplified residues may be most conveniently obtained by a direct process, identical with that which comes into operation in applying to the two given functions of x the method familiarly known under the name of Bezout's abridged method of elimination. Let us call the two given functions U and V, and commence with the case where U and V are of equal dimensions (n) in x. The simplified ith residue will then be a function of $n-i$ dimensions in x, and of i dimensions in respect of each given set of coefficients, and may be taken equal to $V_i.U+U_i.V$, where V_i and U_i are each of $(i-1)$ dimensions in x.

Let

$$U=a_0.x^n+a_1.x^{n-1}+a_2.x^{n-2}+\ldots\ldots+a_n,$$

$$V=b_0.x^n+b_1.x^{n-1}+b_2.x^{n-2}+\ldots\ldots+b_n,$$

we may write in general {m being taken any positive integer not exceeding n},

$$U=(a_0x^m+a_1x^{m-1}+\ldots\ldots+a_m)x^{n-m}+(a_{m+1}x^{n-m-1}+a_{m+2}x^{n-m-2}+\ldots\ldots+a_n)$$

$$V=(b_0x^m+b_1x^{m-1}+\ldots\ldots+b_m)x^{n-m}+(b_{m+1}x^{n-m-1}+b_{m+2}x^{n-m-2}+\ldots\ldots+b_n).$$

Hence

$$\begin{aligned}&(b_0x^m+b_1.x^{m-1}+\ldots\ldots+b_m)U-(a_0x^m+a_1x^{m-1}+\ldots\ldots+a_m)V\\&\quad={}_mK_1x^{n-1}+{}_mK_2.x^{n-2}+{}_mK_3x^{n-3}+\ldots\ldots+{}_mK_n,\end{aligned}\qquad\ldots\ldots\quad(9.)$$

where if we use (r, s) to denote $a_r.b_s-a_s.b_r$ for all values of r and s, we have

${}_mK_1=(0,m+1)$ ${}_mK_2=(0,m+2)+(1,m+1)$ ${}_mK_3=(0,m+3)+(1,m+2)+(2,m+1)$,

and in general ${}_mK_i=\Sigma(r, s)$, the values of r and s admissible within the sign of summation being subject to the two conditions, one the equality $r+s=m+i$, the other the inequality r *less than* i. By giving to m all the different values from 0 to $m-1$ in succession, and calling $b_0x^m+b_1x^{m-1}+\ldots\ldots+b_m$, $a_0x^m+a_1.x^{m-1}+..+a_m$ respectively Q_m and P_m, we have

$$\left.\begin{aligned}Q_0\ .U-P_0\ .V&=K_1\,x^{n-1}+\ K_2x^{n-2}+\ldots\ldots+\ K_n\\Q_1\ .U-P_1\ .V&={}_1K_1.x^{n-1}+\ {}_1K_2x^{n-2}+\ldots\ldots+\ {}_1K_n\\Q_2\ .U-P_2\ .V&={}_2K_1\,x^{n-1}+\ {}_2K_2x^{n-2}+\ldots\ldots+\ {}_2K_n\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\Q_{n-1}.U-P_{n-1}.V&={}_{n-1}K_1\,x^{n-1}+{}_{n-1}K_2x^{n-2}+\ldots\ldots+{}_{n-1}K_n\end{aligned}\right\}\quad\ldots\ldots\quad(10.)$$

The right-hand members of these (n) equations I shall henceforth term the Bezoutians to U and V.

[The determinant formed by arranging in a square the n sets of coefficients of the n Bezoutians, and which I shall term the Bezoutian matrix, gives, as is well known, the Resultant (meaning thereby the Result in its simplest form of eliminating the variables out) of U and V.]

Eliminating dialytically, first x^{n-1} between the first and second, then x^{n-1} and x^{n-2} between the first, second and third, and so on, and finally, all the powers of x between the 1st, 2nd, 3rd, nth of these Bezoutians, and repeating the first of them, we obtain a derived set of (n) equations, the right-hand members of which I shall term the secondary Bezoutians to U and V, this secondary system of equations being

$$\left.\begin{aligned}
&Q_0.U-P_0.V=K_1x^{n-1}+K_2x^{n-2}+K_3x^{n-3}+\ldots+K_n\\
&({}_1K_1Q_0-K_1Q_1)U-({}_1K_1.P_0-K_1.P_1)V=L_1.x^{n-2}+L_2x^{n-3}+\ldots+L_{n-1}\\
&\big(({}_1K_1.{}_2K_2-{}_2K_1.{}_1K_2)Q_0+({}_2K_1.K_2-K_1.{}_2K_2)Q_1+(K_1.{}_1K_2-{}_1K_1.K_2).Q_2\big)U\\
&-\big(({}_1K_1.{}_2K_2-{}_2K_1.{}_1K_2)P_0+({}_2K_1.K_2-K_1.{}_2K_2)P_1+(K_1.{}_1K_2-{}_1K_1.K_2)P_2\big)V\\
&=M_1x^{n-3}+M_2.x^{n-4}+\ldots.+M_{n-2}\\
&\&c.=\&c.
\end{aligned}\right\}\quad(11.)$$

And we can now already without difficulty establish the important proposition, that the successive simplified residues to $\frac{U}{V}$, expanded under the form of an improper continued fraction, abstracting from the algebraical sign (the correctness of which also will be established subsequently), will be represented by the n successive *Secondary Bezoutians* to the system U, V.

For if we write the system of equations (11.) under the general form

$$\mathfrak{S}_i.U-H_i.V=A_i.x^{n-i}+B_ix^{n-i-1}+\&c.,$$

the degree of $\mathfrak{S}_i$ and H_i in x will be that of Q_{i-1} and P_{i-1}, *i. e.* $i-1$; and the dimensions of A_i, B_i, &c., in respect of each set of coefficients is evidently (i); consequently, by virtue of art. (2.), $A_ix^{n-1}+B_ix^{n-2}+$ &c., which is the ith Bezoutian, will (saving at least a numerical factor of a magnitude and algebraical sign to be determined, but which (when proper conventions are made) will be subsequently proved to be $+1$) represent the ith simplified residue to $\frac{U}{V}$*, as was to be shown.

Art. (6.). More generally, suppose U and V to be respectively of $n+e$ and n dimensions in x.

* V is supposed to be taken as the first divisor, and the term residue is used, as hitherto in this paper, throughout in the sense appertaining to the expansion conducted, so as to lead to an *improper* continued fraction, in that sense, in fact, in which it would, more strictly speaking, be entitled to the appellation of *excess* rather than that of *residue*.

Let $$U=a_0.x^{n+e}+a_1.x^{n+e-1}+a_2.x^{n+e-2}+\&c.$$

$$V=b_0.x^n+b_1x^{n-1}+\&c.$$

Making

$$U=(a_0x^{e+m}+a_1.x^{e+m-1}+\&c.+a_{e+m})x^{n-m}+(a_{e+m+1}x^{n-m-1}+\&c.+a_{n+e})$$

$$V=(b_0x^m+b_1x^{m-1}+\ldots+b_m)x^{n-m}+(b_{m+1}x^{n-m-1}+\&c.+b_n),$$

we obtain the equation

$$Q_m.U-P_{e+m}.V={}_mK_1.x^{n+e-1}+{}_mK_2.x^{n+e-2}+\&c.+{}_mK_m, \quad \ldots\ldots\ (12.)$$

where

$$Q_m=(b_0x^m+\ldots+b_m)P_{e+m}=(a_0.x^{e+m}+\ldots+a_{e+m})$$

$${}_mK_1=a_0.b_{m+1};\ {}_mK_2=a_0.b_{m+2}+a_1.b_{m+1};\ \ldots{}_mK_e=a_0.b_{m+e}+a_1.b_{m+e-1}+\&c.+a_e.b_m$$

$${}_mK_{e+1}=a_0.b_{m+e+1}+\&c.+a_{e+1}.b_m-a_{e+m+1}.b_0\ \&c.=\&c.$$

By giving to m every integer value from 0 to $(n-1)$ inclusive, we thus obtain n equations of the form of (12.), each of the degree $n+e-1$ in x, and of one dimension in regard to each set of coefficients.

In addition to these equations we have the (e) equations of the form

$$x^\mu.V=b_0.x^{n+\mu}+b_1.x^{n+\mu-1}+\&c.+b_\mu x^\mu, \quad \ldots\ldots\ (13.)$$

in which μ may be made to assume every value from 0 to $(e-1)$ inclusive, and the left right-hand side of the equation for all such values of μ will remain of a degree in x not exceeding $n+e-1$, the degree of the equations of the system above described. There will thus be (e) equations in which only the (b) set of coefficients appear, and (n) equations containing in every term one coefficient out of each of the two sets.

The total number of equations is of course $n+e$. Between the (e) equations of the second system (13.) and the (r) occurring first in order of the first system (12.), we may eliminate dialytically the $e+r-1$ highest powers of x, and there will thus arise an equation of the form

$$\theta_{r-1}U-\omega_{e+r-1}.V=Lx^{n-r}+L'x^{n-r-1}+\&c.+L\ (14.),$$

where θ_{r-1} and ω_{e+r-1} are respectively of the degrees $r-1$ and $e+r-1$ in x, and $L, L', \ldots (L)$ are of (r) dimensions in the (a) set, and of $(e+r)$ dimensions in the (b) set of coefficients, and consequently $Lx^{n-1}+L'x^{n-r-1}+\ldots+(L)$ must satisfy the conditions necessary and sufficient to prove its being (to a numerical factor *près*) a simplified residue to (U, V).

Thus suppose $$U=a_0.x^4+a_1.x^3+a_2.x^2+a_3.x+a_4$$

$$V=\qquad\qquad b_0x^2+b_1x+b_2.$$

Then, corresponding to the system of which equation (13.) is the type, we have

$$V=b_0.x^3+b_1.x+b_2$$

$$xV=b_0.x^3+b_1.x^2+b_2.x.$$

Again, to form the system of which equation (12.) is the type, we write

$$b_0.U-(a_0x^3+a_1x+a_2)V=b_0(a_3x+a_4)-(a_0x^3+a_1x+a_2)(b_1x+b_2)$$
$$=-a_0b_1.x^3-(a_0b_2+a_1b_1)x^2+(b_0a_3-a_1.b_2-a_2.b_1)x+(b_0a_4-a_2.b_2)$$
$$(b_0x+b_1).U-(a_0x^3+a_1x^2+a_2x+a_3)V=(b_0x+b_1)a_4-(a_0x^3+a_1x^2+a_2x+a_3)b_2$$
$$=-a_0.b_2x^3-a_1b_2x^2+(b_0.a_4-a_2.b_2)x+(b_1a_4-b_2.a_3).$$

Combining the two equations of the first system with the first of the second system, we obtain the first simplified residue $Lx+L'$, where

$$-L=\begin{vmatrix} 0 & b_0 & b_1 \\ b_0 & b_1 & b_2 \\ a_0.b_1 & a_0.b_2+a_1.b_1 & a_1.b_2+a_2.b_1-b_0.a_3 \end{vmatrix}$$

and

$$L'=\begin{vmatrix} 0 & b_0 & b_2 \\ b_0 & b_1 & 0 \\ a_0.b_1 & a_0b_2+a_1b_1 & a_2.b_2-b_0.a_4. \end{vmatrix}$$

By again combining the two equations of the first system with both of the second system, we have the determinant

$$R=\begin{vmatrix} 0 & b_0 & b_1 & b_2 \\ b_0 & b_1 & b_2 & 0 \\ a_0.b_1 & a_0.b_2+a_1.b_1 & a_1.b_2+a_2.b_1-b_1a_3 & a_2.b_2-b_0.a_4 \\ a_0.b_2 & a_1.\ b_2 & a_2.b_2-b_0.a_4 & a_0.b_2-a_4.b_1 \end{vmatrix}$$

which is the last simplified residue, or in other terms, the resultant to the system U, V.

Art. (7.). It is most important to observe that the Bezoutian matrix to two functions of the same degree (n) is a symmetrical matrix, the terms similarly disposed in respect to one of the diagonals being equal.

Thus retaining the notation of art. (5.), so that

$$(0,1)=a\beta-b\alpha \quad (1,2)=b\gamma-c\alpha \quad (2,3)=c\delta-d\gamma$$
$$(0,2)=a\gamma-c\alpha \quad (1,3)=b\delta-d\beta \quad \&c.$$
$$(0,3)=a\delta-d\alpha \quad \&c.$$
$$\&c.$$

&c. &c., when $n=1$ the Bezoutian matrix consists of a single term (0, 1); when $n=2$, it becomes

$$\begin{matrix} (0,1) & (0,2) \\ (0,2) & (1,2) \end{matrix};$$

when $n=3$, it becomes

$$\begin{matrix} (0,1) & (0,2) & (0,3) \\ (0,2) & \begin{pmatrix} (0,3) \\ + \\ (1,2) \end{pmatrix} & (1,3) \\ (0,3) & (1,3) & (2,3) \end{matrix};$$

when $n=4$, it becomes

$$\begin{array}{cccc}
(0,1) & (0,2) & (0,3) & (0,4) \\
(0,2) & \begin{pmatrix}(0,3)\\+\\(1,2)\end{pmatrix} & \begin{pmatrix}(0,4)\\+\\1,3\end{pmatrix} & (1,4) \\
(0,3) & \begin{pmatrix}(0,4)\\+\\(1,3)\end{pmatrix} & \begin{pmatrix}(1,4)\\+\\(2,3)\end{pmatrix} & (2,4) \\
(0,4) & (1,4) & (2,4) & (3,4);
\end{array}$$

when $n=5$, it becomes

$$\begin{array}{ccccc}
(0,1) & (0,2) & (0,3) & (0,4) & (0,5) \\
(0,2) & \begin{pmatrix}(0,3)\\+\\(1,2)\end{pmatrix} & \begin{pmatrix}(0,4)\\+\\(1,3)\end{pmatrix} & \begin{pmatrix}(0,5)\\+\\(1,4)\end{pmatrix} & (1,5) \\
(0,3) & \begin{pmatrix}(0,4)\\+\\(1,3)\end{pmatrix} & \begin{pmatrix}(0,5)\\+\\(2,4)\\+\\(2,3)\end{pmatrix} & \begin{pmatrix}(1,5)\\+\\(2,4)\end{pmatrix} & (2,5) \\
(0,4) & \begin{pmatrix}(0,5)\\+\\(1,4)\end{pmatrix} & \begin{pmatrix}(1,5)\\+\\(2,4)\end{pmatrix} & \begin{pmatrix}(2,5)\\+\\(3,4)\end{pmatrix} & (3,5) \\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5),
\end{array}$$

and so forth. Every such square it is apparent may be conceived as a sort of sloped pyramid, formed by the successive superposition of square layers, which layers possess not merely a simple symmetry about a diagonal (such as is proper to a *multiplication* table), but the higher symmetry (such as exists in an *addition* table), evinced in all the terms in any line of terms parallel to the diagonal transverse to the axis of symmetry being alike*. Thus for $n=5$, the three layers or stages in question will be seen to be, the first—

$$\begin{array}{ccccc}
(0,1) & (0,2) & (0,3) & (0,4) & (0,5) \\
(0,2) & (0,3) & (0,4) & (0,5) & (1,5) \\
(0,3) & (0,4) & (0,5) & (1,5) & (2,5) \\
(0,4) & (0,5) & (1,5) & (2,5) & (3,5) \\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5);
\end{array}$$

the second—

$$\begin{array}{ccc}
(1,2) & (1,3) & (1,4) \\
(1,3) & (1,4) & (2,4) \\
(1,4) & (2,4) & (3,4);
\end{array}$$

and the third—

$$(2,3).$$

In general, when (n) is odd, say $2p+1$, the pyramid will end with a single term

* A square arrangement having this kind of symmetry, viz. such as obtains in the so-called Pythagorean addition table as distinguished from that which obtains in the multiplication table, may be universally called Persymmetric.

$\big(p, (p+1)\big)$, and when even, as $2p$, with a square of 4 terms,

$$\big((p-2), (p-1)\big), \quad \big((p-2), p\big)$$
$$\big((p-2), p\big), \quad \big((p-1), p\big).$$

Each stage may be considered as consisting of three parts, a diagonal set of equal terms transverse to the axis of symmetry, and two triangular wings, one to the left, and the other to the right of this diagonal; the terms in each such diagonal for the respective stages will be

$$(0, n); \quad (1, n-1); \quad \big(2, (n-2)\big); \quad \ldots; \quad \big(p, (p+1)\big),$$

p being $\frac{n}{2}-1$ when n is even, and $\frac{n-1}{2}$ when n is odd.

If we change the order of the coefficients in each of the two given functions, it will be seen that the only effect will be to make the left and right triangular wings to change places, the diagonals in each stage remaining unaltered. The mode of forming these triangles is an operation of the most simple and mechanical nature, too obvious to need to be further insisted on here.

Art. (8.). When we are dealing with two functions of unequal degrees, n and $n+e$, we can still form a square matrix with the coefficients of the two systems of (e) and (n) equations respectively, but this will no longer be symmetrical about a diagonal; it is obvious, however, that if we treat the function of the lower degree, as if it were of the same degree as the other function, which we may do by filling up the vacant places with terms affected with zero coefficients, the symmetry will be recovered; and it is somewhat important (as will appear hereafter) to compare the values of the Bezoutian secondaries as obtained, first in their simplest form by treating each of the two functions as complete in itself, and secondly, as they come out, when that of the functions, which is of the lower degree, is looked upon as a defective form of a function of the same degree as the other. A single example will suffice to make the nature of the relation between the two sets of results apparent.

Take
$$fx = a\,x^4 + b\,x^3 + cx^2 + dx + e$$
$$\varphi x = 0.x^4 + 0.x^3 + \gamma x^2 + \delta x + \varepsilon.$$

The general method of art. (7.) then gives for the Bezoutian matrix

$$\begin{array}{cccc}
0 & a\gamma & a\delta & a\varepsilon \\
a\gamma & \begin{pmatrix} a\delta \\ + \\ b\gamma \end{pmatrix} & \begin{pmatrix} a\varepsilon \\ + \\ b\delta \end{pmatrix} & b\varepsilon \\
a\delta & \begin{pmatrix} a\varepsilon \\ + \\ b\delta \end{pmatrix} & \begin{pmatrix} b\varepsilon \\ + \\ c\delta - d\gamma \end{pmatrix} & c\varepsilon - e\gamma \\
a\varepsilon & b\varepsilon & c\varepsilon - e\gamma & d\varepsilon - e\delta.
\end{array}$$

We shall not affect the value either of the complete determinant, or of any of the minor determinants appertaining to the above matrix, by subtracting the second line of terms, each increased in the ratio of $b : a$ from the first line of terms respectively;

the matrix so modified becomes

$$\begin{array}{llll} 0\,; & a\gamma\,; & a\delta\,; & a\varepsilon \\ a\gamma\,; & a\delta\,; & a\varepsilon\,; & 0 \\ a\delta\,; & a\varepsilon+b\delta\,; & \begin{pmatrix} b\varepsilon \\ + \\ c\delta-d\gamma \end{pmatrix}; & c\varepsilon-e\gamma \\ a\varepsilon\,; & b\varepsilon\,; & c\varepsilon-e\gamma\,; & d\varepsilon-e\delta. \end{array}$$

Again, adopting the method of art. (6.), we should obtain the matrix

$$\begin{array}{llll} 0\,; & \gamma\,; & \delta\,; & \varepsilon \\ \gamma\,; & \delta\,; & \varepsilon\,; & 0 \\ \delta\,; & a\varepsilon-b\delta\,; & \begin{pmatrix} b\varepsilon \\ + \\ c\delta-d\gamma \end{pmatrix}; & c\varepsilon-e\gamma \\ a\varepsilon\,; & b\varepsilon\,; & c\varepsilon-e\gamma\,; & d\varepsilon-e\delta. \end{array}$$

Hence it is apparent that the secondary Bezoutians obtained by the symmetrizing method will differ from those obtained by the unsymmetrical method by a constant factor a^3; and so in general it may readily be shown that the secondary Bezoutians, by the use of the symmetrizing method, will each become affected with a constant irrelevant factor a^ω, where (ω) is the difference of the degrees of the two functions, and (a) the leading coefficient of the higher one of the two. When (a) is taken unity, the Bezoutian secondaries, as obtained by either method, will of course be identical.

Art. (9.). There is another method* of obtaining the simplified residues to anytwo functions U and V of the degrees n and $n+e$ respectively, which, although less elegant, ought not to be passed over in silence. This method consists in forming the identical equations (of which for greater brevity the right-hand members are suppressed).

$$\begin{aligned} \mathrm{V} &= \&c. \\ x\mathrm{V} &= \&c. \\ &\vdots \\ x^{e-1}.\mathrm{V} &= \&c. \\ \mathrm{U} &= \&c. \\ x^e.\mathrm{V} &= \&c. \\ x.\mathrm{U} &= \&c. \\ x^{e+1}.\mathrm{V} &= \&c. \\ x^2.\mathrm{U} &= \&c. \\ x^{e+2}.\mathrm{V} &= \&c. \\ \&c. &= \&c. \\ x^{n-1}.\mathrm{U} &= \&c. \\ x^{e+n-1}.\mathrm{V} &= \&c. \end{aligned}$$

* Originally given by myself in the London and Edinburgh Philosophical Magazine, as long ago as 1839 or 1840; and some years subsequently in unconsciousness of that fact, reproduced by my friend Mr. Cayley, to whom the method is sometimes erroneously ascribed, and who arrived at the same equations by an entirely different circle of reasoning.

If we equate the right-hand members of $(e+2\iota)$ of the above equations to zero, and then eliminate dialytically the several powers of x from $x^{n+e+\iota-1}$ to $x^{n-\iota+1}$ (both inclusive), the result of this process will evidently be of $(e+\iota)$ dimensions in respect of the coefficients in V, and of ι dimensions in respect of the coefficients in U; and of the degree $x^{n-\iota}$ in x it will also be of the form

$$(A+Bx+\ldots Lx^{e-1})U+(F+Gx+\ldots+Qx^{e+\iota-1}),$$

and by virtue of art. (2.) must consequently be the ιth simplified residue to the system U, V.

Art. (10.). The most general view of the subject of expansion by the method of continued division, consists in treating the process as having reference solely to the two systems of coefficients in U and V, which themselves are to be regarded in the light of generating functions. To carry out this conception, we ought to write

$$U=a_0+a_1.y+a_2.y^2+a_3y^3+\&c.\ ad\ inf.$$
$$V=b_0+b_1.y+b_2.y^2+b_3y^3+\&c.\ ad\ inf.,$$

and might then suppose the process of successive division applied to U and V, so as to obtain the successive equations

$$\begin{aligned} U-M_1V+R_1&=0\\ V-M_2R_1+R_2&=0\\ R_1-M_3R_2+R_3&=0\\ &\&c.\ \&c., \end{aligned}$$

M_1, M_2, M_3, &c. being each severally of any degree whatever in y, and in general the degree of y in M_ι being any given arbitrary function $\varphi(\iota)$ of ι. The values of the coefficients of the residues R_1, R_2, R_3 ..., or of these forms simplified by the rejection of detachable factors, becomes then the distinct object of the inquiry, and will, of course, depend only upon the coefficients in P and Q and the nature of the arbitrary continuous or discontinuous function $\varphi(\iota)$, which regulates the number of *steps* through which each successive process of division is to be pursued. Following out this idea in a particular case, if we again reduce to our two initial functions the forms previously employed, and write

$$U=a_0.x^n+a_1.x^{n-1}+\&c.$$
$$V=b_0.x^n+b_1.x^{n-1}+\&c.;$$

and if, instead of making, according to the more usual course of proceeding, the divisions proceed first through one step and ever after through two steps at a time which is tantamount to making $\varphi 1=1$ $\varphi(1+\omega)=2$, we push each division through one step only at a time, and no more (so that in fact $\varphi(i)$ is always 1), we shall have

$$\begin{aligned} U-m_1.\ V+R_1&=0\\ V-m_2x.\ R_1+R_2&=0\\ R_1-m_3.\ R_2+R_3&=0\\ R_2-m_4.x.R_3+R_4&=0\\ &\&c.\ \&c., \end{aligned}$$

m_1, m_2, m_3, &c. being functions of the coefficients only of U and V; and it is not without interest to observe (which is capable of an easy demonstration) that the simplified residues contained in R_1, R_2, &c., found according to this mode of development, will be the successive dialytic resultants obtained by eliminating the $(i-1)$th highest powers of x between the i first of the system of the annexed equations (supposed to be expressed in terms of x)

$$\begin{aligned} U&=0\\ V&=0\\ x.U&=0\\ x.V&=0\\ x^2.U&=0\\ x^2.V&=0\\ &\&c.\ \&c.\\ x^{n-1}.U&=0\\ x^{n-1}.V&=0. \end{aligned}$$

If we combine together $2i+1$ of the above equations, the highest power of x entering on the left-hand side will be x^{n+i}, and we shall be able to eliminate $2i$ of these factors, leaving x^{n-i} the highest power remaining uneliminated. If we take $2i$, *i. e.* i pairs of the equations, the highest power of x appearing in any of them will be x^{n+i-1}, and we shall be able to eliminate between them so as still to leave $x^{n+i-1-(2i-1)}$, *i. e.* x^{n-i} as before, the highest power of x remaining uneliminated; and it will be readily seen that such of the simplified residues corresponding to this mode of development as occupy the odd places in the series of such residues, will be identical with the successive simplified residues resulting from the ordinary mode of developing $\frac{U}{V}$ under the form of a continued fraction.

Art. (11.). It has been shown that the simplified residues of fx and φx resulting from the process of continued division are identical in point of form with the secondary Bezoutians of these functions, but it remains to assign the numerical relations between any such residue and the corresponding secondary.

To determine this numerical relation, it will of course be sufficient to compare the magnitude of the coefficient of any one power of x in the one, with that of the same power in the other; and for this purpose I shall make choice of the leading coefficients in each. In what follows, and throughout this paper, it will always be understood that in calculating the determinant corresponding to any square the product of the terms situated in the diagonal descending from left to right will always be taken with the positive sign, which convention will serve to determine the sign of all the other products entering into such determinant. Now adopting the umbral notation for determinants*, we have, by virtue of a much more general theorem for compound

* See London and Edinburgh Philosophical Magazine, April 1851.

determinants, the following identical equation :—

$$\begin{matrix} a_1a_2a_3\ldots a_{m-1} \\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m-1}\end{matrix} \times \begin{matrix} a_1a_2a_3\ldots a_{m+1} \\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m+1}\end{matrix}$$

$$= \begin{pmatrix} a_1a_2a_3\ldots a_{m-1}a_m \\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m-1}\alpha_m\end{pmatrix} \times \begin{pmatrix} a_1a_2\ldots a_{m-1}a_{m+1} \\ \alpha_1\alpha_2\ldots\alpha_{m-1}\alpha_{m+1}\end{pmatrix}$$

$$- \begin{pmatrix} a_1a_2a_3\ldots a_{m-1}a_m \\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m-1}\alpha_{m+1}\end{pmatrix} \times \begin{pmatrix} a_1a_2\ldots a_{m-1}a_{m+1} \\ \alpha_1\alpha_2\ldots\alpha_{m-1}\alpha_m\end{pmatrix}$$

and consequently

$$\begin{matrix} a_1a_2a_3\ldots a_{m-1} \\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m-1}\end{matrix} \times \begin{matrix} a_1a_2a_3\ldots a_{m-1}\ a_m.a_{m+1} \\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m-1}.\alpha_m.a_{m+1}\end{matrix}$$

$$= \begin{pmatrix} a_1a_2a_3\ldots a_{m-1}a_m \\ \alpha_1\alpha_2\alpha_3\ldots\alpha_{m-1}\alpha_m\end{pmatrix} \times \begin{pmatrix} a_1a_2\ldots a_{m-1}a_{m+1} \\ \alpha_1\alpha_2\ldots\alpha_{m-1}\alpha_{m+1}\end{pmatrix}$$

$$- \begin{pmatrix} a_1a_2\ldots a_{m-1}\ a_m \\ \alpha_1\alpha_2\ldots\alpha_{m-1}.\alpha_{m+1}\end{pmatrix}^2$$

and consequently when

$$\begin{Bmatrix} a_1a_2\ldots a_{m-1}\ a_m \\ \alpha_1\alpha_2\ldots\alpha_{m-1}.\alpha_{m+1}\end{Bmatrix} = 0$$

$\begin{matrix} a_1a_2\ldots a_{m-1} \\ \alpha_1\alpha_2\ldots\alpha_{m-1}\end{matrix}$ and $\begin{matrix} a_1a_2\ldots a_{m-1}.a_ma_{m+1} \\ \alpha_1\alpha_2\ldots\alpha_{m-1}.a_m\alpha_{m+1}\end{matrix}$

will have different algebraical signs, it being of course understood that all the quantities entering into the determinants thus *umbrally* represented above are supposed to be real quantities. This theorem, translated into the ordinary language of determinants, may be stated as follows :—Begin with any square of terms whether symmetrical or otherwise, say of r lines and r columns; let this square be bordered laterally and longitudinally by the same r new quantities symmetrically disposed in respect to one of the diagonals, the term common to the superadded line and column being filled up with any quantity whatever; we thus obtain a square of $(r+1)$ lines and columns; let this be again bordered laterally and longitudinally by $(r+1)$ quantities symmetrically disposed above the same diagonal as that last selected, the place in which this new line and column meet being also filled up with any arbitrary quantity; and proceeding in this manner, let the determinants corresponding to the square matrices thus formed be called D_{r-1}, D_r, D_{r+1}, D_{r+2}, ... this series of quantities will possess the property, that no term in it can vanish without the terms on either side of that so vanishing having contrary signs. Thus if we begin with a square consisting of one single term, we may suppose that by accretions formed after the above rule it has been developed into the square (M) below written, and which of course may be

indefinitely extended:—

$$\begin{matrix} a & l & m & p & s \\ l & b & n & q & t \\ m & n & c & r & u \\ p & q & r & d & v \\ s & t & u & v & e \end{matrix} ; \quad \ldots\ldots\ldots \quad \text{(M.)}$$

r here begins with the value (1), and D_0, D_1, D_2, D_3, D_4, D_5 will represent the progression,

$$1 ; \quad a ; \quad \begin{matrix} a & l \\ l & b \end{matrix} ; \quad \begin{matrix} a & l & m \\ l & b & n \\ m & n & c \end{matrix} ; \quad \begin{matrix} a & l & m & p \\ l & b & n & q \\ m & n & c & r \\ p & q & r & d \end{matrix} ; \quad \begin{matrix} a & l & m & p & s \\ l & b & n & q & t \\ m & n & c & r & u \\ p & q & r & d & v \\ s & t & u & v & e \end{matrix} \ldots \quad \text{(II.)}$$

so if we use the matrix

$$\begin{matrix} a & l & m & p & s \\ l' & b & n & q & t \\ m & n & c & r & u \\ p & q & r & d & v \\ s & t & u & v & e \end{matrix}$$

the determinants D_1, D_2, D_3, D_4, D_5 representing

$$a ; \quad \begin{matrix} a & l \\ l' & b \end{matrix} ; \quad \begin{matrix} a & l & m \\ l' & b & n \\ m & n & c \end{matrix} ; \quad \begin{matrix} a & l & m & p \\ l' & b & n & q \\ m & n & c & r \\ p & q & r & d \end{matrix} ; \text{ \&c.}$$

will possess the property in question; the line and column l, b; l', b not being identical, the first determinant D_0 representing (1) must not be included in the progression.

We shall have occasion to use this theorem as applicable to the case of a matrix symmetrical throughout, and we may term the progression (II) above written a progression of the successive principal determinants about the axis of symmetry of the square matrix (M), and so in general. Now it is obvious that the leading coefficients of the successive Bezoutian secondaries are the successive principal determinants about the axis of symmetry of the Bezoutian squares; they will therefore have the property which has been demonstrated of such progressions; to wit, if the first of them vanishes, the second will have a sign contrary to that of $+1$; if the second vanishes, the third will have a sign contrary to that of the first, and so on.

Art. (12.). Now let fx and φx be any two algebraical functions of x with the leading coefficients in each, for greater simplicity supposed positive: and in the course of developing $\frac{\varphi x}{fx}$ under the form of an improper continued fraction by the common process of successive division, let any two consecutive residues (the word residue being

used in the same conventional sense as employed throughout) be

$$Ax+Bx^{i-1}+Cx^{i-2}+\&c.$$
$$B'x^{i-1}+C'x^{i-2}+D'x^{i-3}+\&c.$$

The residue next following, obtained by actually performing the division and duly changing the sign of the remainder will be

$$\left\{\left(\frac{AD}{B'}-C\right)-\left(\frac{AC'}{B'}-B\right)\frac{C'}{B'}\right\}x^{i-2}+\&c.,$$

which is of the form

$$\frac{1}{B'^2}\left\{B'M-AC'^2\right\}x^{i-2}+\&c.$$

Thus the leading coefficients in the complete unreduced residues will be

$$A;\ B';\ \frac{1}{B'^2}\left\{B'M-AC'^2\right\},$$

and when reduced by the expulsion of the allotrious factor will become A; B'; $B'.M-AC'^2$, and consequently, when B' the leading coefficient of one of the simplified residues vanishes, the leading coefficients of the residues immediately preceding and following that one will have contrary signs.

First, let

$$fx=ax^n+bx^{n-1}+\&c.,\quad \varphi x=\alpha x^n+\beta x^{n-1}+\&c.$$

As regards the numerical ratio of each Bezoutian secondary to the corresponding simplified residue, it has been already observed that there are always unit coefficients in the latter of these, and the same is obviously true of the former; hence if we call the progression of the leading coefficients of the simplified residues

$$R_1;\ R_2;\ R_3;\ R_4,\ \&c.,$$

and that of the leading coefficients of the Bezoutian secondaries

$$B_1;\ B_2;\ B_3;\ B_4,\ \&c.,$$

we have

$$B_1=\pm R_1\quad B_2=\pm R_2\quad B_3=\pm R_3\quad B_4=\pm R_4,\ \&c.$$

It may be proved by actual trial that $B_1=R_1$ and $B_2=R_2$. Moreover, since the signs are invariable, and do not depend upon the values of the coefficients, we may suppose $B_2=0$ (which may always be satisfied by real values of the quantities, of which B_2 is a function); we shall also, therefore, have $R_2=0$, and consequently B_3 has the opposite sign to that of B_1, and R_3 the opposite sign to that of R_1, which is equal to B_1; hence when $B_2=0$, B_3 and R_3 are equal, and consequently are always equal; in like manner we can prove that R_4 and B_4 have the same sign when R_3 and B_3 vanish, and consequently are always equal, and so on *ad libitum*, which proves that the series $B_1, B_2, \ldots B_n$ is identical with the series $R_1, R_2, \ldots R_n$, and consequently that the Bezoutian secondaries are identical in form, magnitude and algebraical sign with the simplified residues. Secondly, when fx and φx are not of the same degree, it has been shown that the secondaries formed from the non-

symmetrical matrix corresponding to this case will be the same as those formed from the *symmetrical* matrix corresponding to fx and $\Phi(x)$ (where Φx is $\varphi(x)$ treated by aid of evanescent terms, as of the same degree as fx), with the exception merely of a constant multiplier (a power of the leading coefficient of fx) being introduced into each secondary. By aid of this observation, the proposition established for the case of two functions of the same degree may be readily seen to be capable of being extended, from the case of f and φ being of the equal dimensions in x, to the general case of their dimensions being any whatever.

Art. (13.). Before closing this section, it may be well to call attention to the nature of the relation which connects the successive residues of fx and φx with these functions themselves, and with the improper continued fractional form into which $\frac{\varphi x}{fx}$ is supposed to be developed in the process of obtaining these residues.

If φx be of n degrees, and fx of $n+e$ degrees in (n), we shall have

$$\frac{\varphi x}{fx}=\frac{1}{Q_1-}\frac{1}{q_2-}\frac{1}{q_3-}\cdots\frac{1}{q_n},$$

where Q_1 may be supposed to be a function of x of the degree (e), and $q_2, q_3, \ldots q_n$ are all linear functions of x; the total number of the quotients $Q_1, q_2, \ldots q_n$ being of course (n) when the process of continued division is supposed to be carried out until the last residue is zero. Upon this supposition the last but one residue is a constant, the preceding one a function of x of the first degree, the one preceding that a function of x of the second degree, and so on.

Let us call the residue of the degree ϵ in x, $\mathfrak{S}_\epsilon$; it will readily be seen that the successive complete residues arranged in an ascending order will be

$$\mathfrak{S}_0, \mathfrak{S}_0.q_n, \mathfrak{S}_0(q_{n-1}.q_n-1), \quad \mathfrak{S}_0(q_{n-2}.q_{n-1}.q_n-q_{n-2}-q_n); \text{ \&c.},$$

being in the ratios of the quantities

$$1; \quad q_n; \quad q_{n-1}-\frac{1}{q_n}; \quad q_{n-2}-\frac{1}{q_{n-1}}\frac{1}{q_n}; \text{ \&c.}$$

Again, we shall have in general

$$A_\iota f-L_\iota\varphi=\mathfrak{S}_\iota \quad \ldots\ldots\ldots\ldots \quad (15.)$$

A_ι being an integral function of x of the degree $n-\iota-1$, and L_ι an integral function of x of the degree $(n+e)-\iota-1$; and it is easy to see that the successive convergents to the continued fraction—

$$\frac{1}{Q_1-}\frac{1}{q_2-}\frac{1}{q_3-} \text{ \&c.}$$

have their respective numerators and denominators identical with those of the fractions

$$\frac{A_{n-1}}{L_{n-1}}, \frac{A_{n-2}}{L_{n-2}}, \frac{A_{n-3}}{L_{n-3}} \text{ \&c.}$$

Adopting the language which I have frequently employed elsewhere, I call $\mathfrak{S}_\iota$ a

syzygetic function, or more briefly, a conjunctive of f and φ, and Λ_i and L_i may be termed the syzygetic factors to $\mathfrak{S}_i$ so considered. If we divide each term of the equation (15.) by the allotrious factor (M), we have

$$\frac{\Lambda_i}{M}f-\frac{L_i}{M}\varphi=R_i,$$

where R_i is the ith simplified residue to (f, φ); and if we call $\frac{\Lambda_i}{M}=\tau_i$, and $\frac{L_i}{M}=t_i$, so as to obtain the equation

$$\tau_i.f-t_i.\varphi=R_i, \quad \ldots\ldots\ldots\ldots \quad (16.)$$

we see that $\frac{\tau_i}{t_i}$, the fraction formed by the component factors to any simplified residue of (f, φ), will be identical in value (although no longer in its separate terms) with one of the corresponding convergents to $\frac{\phi}{f}$, exhibited under the form of an improper continued fraction. I shall in the next section show how, not only the successive simplified residues, but also the component syzygetic factors of each of them, and consequently the successive convergents, may be expressed in terms of the roots of the two given functions.

Since the preceding section was composed the valuable memoir of the lamented JACOBI, entitled "De Eliminatione Variabilis è duabus Equationibus Algebraicis," CRELLE, vol. xvi., has fallen under my notice. That memoir is restricted to the consideration of two equations of the same degree, and the principal results in this section as regards the Bezoutic square and the allotrious factors applicable to that case will be found contained therein. The mode of treatment however is sufficiently dissimilar to justify this section being preserved unaltered under its original form.

SECTION II.

On the general solution in terms of the roots of any two given algebraical functions of x *of the syzygetic equation, which connects them with a third function, whose degree in* (x) *is given, but whose form is to be determined.*

Art. (14.). Let f and φ be two given functions in x of the degrees m and n respectively in x, and for the sake of greater simplicity let the coefficients of the highest power of x in f and φ be each taken unity, and let it be proposed to solve the syzygetic equation

$$\tau_i.f-t_i.\varphi+\mathfrak{S}_i=0, \quad \ldots\ldots\ldots\ldots \quad (17.)$$

where $\mathfrak{S}_i$ is given only in the number of its dimensions in x, which I suppose to be (i); but the forms of τ_i, t_i, $\mathfrak{S}_i$ are all to be determined in terms of $h_1, h_2 \ldots h_m$ the roots of f and $\eta_1, \eta_2, \ldots \eta_n$ the roots of φ.

I shall begin with finding $\mathfrak{S}_i$; and before giving a more general representation of $\mathfrak{S}_i$, I propose now to demonstrate that we may make

$$\mathfrak{S}_i=\Sigma\{P_{\eta_1,\eta_2\ldots\eta_i}\times(x-h_{\eta_1})(x-h_{\eta_2})\ldots(x-h_{\eta_i})\}, \quad \ldots\ldots \quad (18.)$$

where $P_{q_1, q_2 \ldots q_i}$ is used to denote

$$\left\{\begin{array}{l} (h_{q_{i+1}}-\eta_1)(h_{q_{i+1}}-\eta_2)\ \ldots\ (h_{q_{i+1}}-\eta_m) \\ \times(h_{q_{i+2}}-\eta_1)(h_{q_{i+2}}-\eta_2)\ \ldots\ (h_{q_{i+2}}-\eta_m) \\ \times(h_{q_{i+3}}-\eta_1)(h_{q_{i+3}}-\eta_2)\ \ldots\ (h_{q_{i+3}}-\eta_m) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \times(h_{q_m}-\eta_1)(h_{q_m}-\eta_2)\ \ldots\ (h_{q_m}-\eta_m) \end{array}\right\} \mathrm{R}.(h_{q_1}\, h_{q_2} \ldots h_{q_i}),$$

$\mathrm{R}(h_{q_1}\, h_{q_2} \ldots h_{q_i})$ denoting any rational symmetrical form of function whatever of the quantities preceded by the symbol R, and $q_1\, q_2 \ldots q_i\, q_{i+1} \ldots q_m$ being any permutation of the m indices $1, 2, \ldots m$.

Suppose $f=0$ and $\phi=0$, then x is equal to one of the series of roots

$$h_1\, h_2 \ldots h_m,$$

and also to one of the series of roots

$$\eta_1\, \eta_2 \ldots \eta_n.$$

Suppose then that $x=h_a=\eta_\nu$,

and consider any term of $\mathfrak{S}_i$.

If in any such term (a) is found in the series $q_1\, q_2 \ldots q_i$, then

$$(x-h_{q_1})(x-h_{q_2}) \ldots (x-h_{q_i})=0.$$

But if not, then (a) must be found in the complementary series $h_{q_{i+1}}, h_{q_{i+2}}, \ldots, h_{q_m}$, and consequently $P_{q_1 q_2 \ldots q_i}$ will contain a factor $h_a-\eta_\nu$ and $P_{q_1 q_2 \ldots q_i}=0$; in every case therefore

$$P_{q_1 q_2 \ldots q_i} \times (x-h_{q_1})(x-h_{q_2}) \ldots (x-h_{q_i})=0,$$

and therefore $\mathfrak{S}_i$, as expressed in equation (18.) is a syzygetic function of f and ϕ; accordingly we have found a function of the ith degree in x, and of course expressible by calculating the symmetric functions as a function only of x and of the coefficients of f and ϕ, which will satisfy the equation

$$\tau_i . f - t_i . \phi + \mathfrak{S}_i = 0.$$

[It will be remembered that by virtue of art. (2) we know *à priori* that all the values of $\mathfrak{S}_i$ satisfying this equation are identical, save as to an allotrious factor, which is a function only of the coefficients in f and ϕ.] It is clear that we may interchange the h and η, m and n, and thus another representation of a value of $\mathfrak{S}_i$ satisfying the equation (17.) will be

$$\mathfrak{S}_i = \Sigma \mathrm{R}(\eta_{q_1} \eta_{q_2} \ldots \eta_{q_i}) \times \left\{\begin{array}{l} (\eta_{q+1}-h_1)(\eta_{q+2}-h_2) \ldots (\eta_{q+1}-h_m) \\ (\eta_{q+2}-h_1)(\eta_{q+2}-h_2) \ldots (\eta_{q+2}-h_m) \\ (\eta_{q+3}-h_1)(\eta_{q+3}-h_2) \ldots (\eta_{q+3}-h_m) \\ \ldots\ldots\ldots\ldots\ldots \\ (\eta_{q_m}-h_1)\ (\eta_{q_m}-h_2) \ldots (\eta_{q_m}-h_m) \end{array}\right\} (x-\eta_{q_1})(x-\eta_{q_2}) \ldots (x\quad \eta_{q_i}).$$

Art. (15.). If we employ in general the condensed notation

$$\begin{bmatrix} l, m, n, \ldots p \\ \lambda, \mu, \ldots \nu \end{bmatrix}$$

to denote the product of the differences resulting from the subtraction of each of the quantities $\lambda, \mu, \ldots \nu$ in the lower line from all of those in the upper line $l, m, n, \ldots p$, the two values above given for $\mathfrak{S}_i$ may be written under the respective forms

$$\Sigma R(h_{q_1} h_{q_2} \ldots h_{q_i}) \cdot \begin{bmatrix} h_{q_{i+1}} h_{q_{i+2}} \ldots h_{q_m} \\ \eta_1 \quad \eta_2 \quad \ldots \eta_n \end{bmatrix} (x-h_{q_1})(x-h_{q_2}) \ldots (x-h_{q_i})$$

and

$$\Sigma R(\eta_{\xi_1} \eta_{\xi_2} \ldots \eta_{\xi_i}) \cdot \begin{bmatrix} \eta_{\xi_{i+1}} \eta_{\xi_{i+2}} \ldots \eta_{\xi_n} \\ h_1 \quad h_2 \quad \ldots h_m \end{bmatrix} \times (x-\eta_{\xi_1})(x-\eta_{\xi_2}) \ldots (x-\eta_{\xi_i})$$

in each of which equations disjunctively and in some order of relation each with each

$$q_1, q_2, q_3 \ldots, q_m = 1, 2, 3, \ldots, m,$$

and

$$\xi_1, \xi_2, \xi_3 \ldots, \xi_n = 1, 2, 3, \ldots, n.$$

These two forms are only the two extremities of a scale of forms all equally well adapted to express $\mathfrak{S}_i$; for let v and ν be any two integers so taken as to satisfy the equation

$$v + \nu = i,$$

and let $R(\overline{\cdots\cdots}; \overline{\cdots\cdots})$, where the dots denote any quantities whatever, be used to denote a rational form of function which remains unaltered in value when any two of the quantities under each and either (the same one) of the two bars are mutually interchanged, then we may write

$$\mathfrak{S}_i = \Sigma \left\{ R(\overline{h_{q_1} h_{q_2} \ldots h_{q_v}}; \overline{\eta_{\xi_1} \eta_{\xi_2} \ldots \eta_{\xi_\nu}}) \times \begin{bmatrix} h_{q_{v+1}} h_{q_{v+2}} \ldots h_{q_m} \\ \eta_{\xi_{\nu+1}} \eta_{\xi_{\nu+2}} \ldots \eta_{\xi_n} \end{bmatrix} \times (x-h_{q_1})(x-h_{q_2}) \ldots (x-h_{q_v}) \times (x-\eta_{\xi_1})(x-\eta_{\xi_2}) \ldots (x-\eta_{\xi_\nu}) \right\} \quad . \quad . \quad (19.)$$

For if, as above, we suppose $x = h_a = \eta_\alpha$, any term of $\mathfrak{S}_i$ in which $q_1, q_2, \ldots q_v$ comprise among them h_a, or in which $\xi_1 \xi_2 \ldots \xi_\nu$ comprise among them η_α, will vanish by virtue of the factors $(x-h_{q_1})(x-h_{q_2}) \ldots (x-h_{q_v}) \times (x-\eta_{\xi_1})(x-\eta_{\xi_2}) \ldots (x-\eta_{\xi_\nu})$; but if neither h_a nor η_α is so comprised, then h_a must be one of the terms in the complementary series $q_{v+1}, q_{v+2}, \ldots q_m$, and η_α one of the terms in the complementary series $\xi_{\nu+1}, \xi_{\nu+2}, \ldots \xi_n$, and therefore one of the quantities $h_{q_{v+1}}, h_{q_{v+2}}, \ldots h_{q_m}$ will equal one of the quantities $\eta_{\xi_{\nu+1}}, \eta_{\xi_{\nu+2}}, \ldots \eta_{\xi_n}$, and consequently the term of $\mathfrak{S}_i$ in question will vanish by virtue of the factor $\begin{bmatrix} h_{q_{v+1}} h_{q_{v+2}} \ldots h_{q_m} \\ \eta_{\xi_{\nu+1}} \eta_{\xi_{\nu+2}} \ldots \eta_{\xi_n} \end{bmatrix}$ vanishing. In either case therefore every term included within the sign of summation vanishes when $x = h_a = \eta_\alpha$, *i. e.* whenever $f(x) = 0$ and $\phi = (x) = 0$. Hence $\mathfrak{S}_i$, as given by equation (19.), will satisfy the syzygetic equation

$r.f-t_r.\varphi+\mathfrak{S}_r=0$ for all values of v and υ which make $v+\upsilon=\iota$, and for all symmetrical forms of the function denoted by the symbol $\mathrm{R}(\ldots\ldots;\ \ldots\ldots)$.

Art. (16.). I shall now proceed to show how to assign the arbitrary function whose form is denoted by this symbol in such a manner as to make $\mathfrak{S}_r$ become identical with a simplified residue to f and φ. To this end I take for $\mathrm{R}(h_{q_1}h_{q_2}\ldots h_{q_\iota};\ k_{\xi_1}k_{\xi_2}\ldots k_{\xi_\upsilon})$ the value

$$\mathrm{R}=\frac{\begin{bmatrix} h_{q_1}h_{q_2}\ldots h_{q_v} \\ \eta_{\xi_1}\eta_{\xi_2}\ldots\eta_{\xi_\upsilon}\end{bmatrix}}{\begin{bmatrix} h_{q_1} & h_{q_2} & \ldots h_{q_v} \\ h_{q_{v+1}} & h_{q_{v+2}} & \ldots h_{q_m}\end{bmatrix}\times\begin{bmatrix} k_{\xi_1} & k_{\xi_2} & \ldots k_{\xi_\upsilon} \\ k_{\xi_{\upsilon+1}} & k_{\xi_{\upsilon+2}} & \ldots k_{\xi_n}\end{bmatrix}}, \quad \ldots\ldots \quad (20.)$$

we shall then have

$$\mathfrak{S}_r=\Sigma\frac{\begin{bmatrix} h_{q_1}h_{q_2}\ldots h_{q_v}\end{bmatrix}\times\begin{bmatrix} h_{q_{v+1}}h_{q_{v+2}}\ldots h_{q_m}\end{bmatrix}}{\begin{bmatrix} h_{q_1} & h_{q_2} & \ldots h_{q_v} \\ h_{q_{v+1}} & h_{q_{v+2}} & \ldots h_{q_m}\end{bmatrix}\times\begin{bmatrix} \eta_{\xi_1} & \eta_{\xi_2} & \ldots & \eta_{\xi_\upsilon} \\ \eta_{\xi_{\upsilon+1}} & \eta_{\xi_{\upsilon+2}} & \ldots & \eta_{\xi_n}\end{bmatrix}}\{(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_v})\}\{(x-\eta_{\xi_1})(x-\eta_{\xi_2})\ldots(x-\eta_{\xi_\upsilon})\} \quad (21.)$$

I shall first show this sum of fractions is in substance an integral function of the quantities $h_1h_2\ldots h_m$; $k_1k_2\ldots k_m$. For greater conciseness write in general $x-h=\mathrm{E}$, $x-\eta=\mathrm{H}$, we have then, since $h-\eta=\mathrm{H}-\mathrm{E}$, $h_{q_1}-h_{q_2}=\mathrm{E}_{q_1}-\mathrm{E}_{q_2}$, $\eta_{\xi_1}-\eta_{\xi_2}=\mathrm{H}_{\xi_1}-\mathrm{H}_{\xi_2}$,

$$\mathfrak{S}_r=\Sigma\left\{\frac{\begin{bmatrix} \mathrm{H}_{\xi_1}\mathrm{H}_{\xi_2}\ldots\mathrm{H}_{\xi_\upsilon}\end{bmatrix}\times\begin{bmatrix} \mathrm{H}_{\xi_{\upsilon+1}}\mathrm{H}_{\xi_{\upsilon+2}}\ldots\mathrm{H}_{\xi_n}\end{bmatrix}}{\begin{bmatrix} \mathrm{E}_{q_1} & \mathrm{E}_{q_2} & \ldots\mathrm{E}_{q_v} \\ \mathrm{E}_{q_{v+1}} & \mathrm{E}_{q_{v+2}} & \ldots\mathrm{E}_{q_m}\end{bmatrix}\times\begin{bmatrix} \mathrm{H}_{\xi_1} & \mathrm{H}_{\xi_2} & \ldots\mathrm{H}_{\xi_\upsilon} \\ \mathrm{H}_{\xi_{\upsilon+1}} & \mathrm{H}_{\xi_{\upsilon+2}} & \ldots\mathrm{H}_{\xi_n}\end{bmatrix}}.\mathrm{E}_{q_1}\mathrm{E}_{q_2}\ldots\mathrm{E}_{q_m}\right\}. \quad \ldots\ldots \quad (22.)$$

On reducing the fractions contained within the sign of summation to a common denominator, $\mathfrak{S}_r$ will take the form $\frac{\mathrm{N}}{\mathrm{D}.\Delta}$, where D will be the product of the $m.\frac{m-1}{2}$ differences of $\mathrm{E}_1, \mathrm{E}_2, \ldots\mathrm{E}_m$ subtracted each from each, and Δ the corresponding product of the differences *inter se* of $\mathrm{H}_1, \mathrm{H}_2, \ldots\mathrm{H}_m$.

Hence, unless the sum in question is an integral function of the E's and H's, it will become infinite when any two of the E series, or any two of the H series of quantities are made equal. Suppose now $\mathrm{E}_1=\mathrm{E}_2$; the terms in (22.) which contain $\mathrm{E}_1-\mathrm{E}_2$ in the denominator will evidently group themselves into pairs of the respective forms,

$$\frac{(\mathrm{E}_1.\mathrm{E}_{q_3}\ldots\mathrm{E}_{q_v})\times(\mathrm{H}_{\xi_1}\mathrm{H}_{\xi_2}\ldots\mathrm{H}_{\xi_\upsilon})\times\begin{bmatrix} \mathrm{E}_1 & \mathrm{E}_{q_3}\ldots\mathrm{E}_{q_v} \\ \mathrm{H}_{\xi_1}\mathrm{H}_{\xi_2}\ldots\mathrm{H}_{\xi_\upsilon}\end{bmatrix}\times\begin{bmatrix} \mathrm{E}_2 & \mathrm{E}_{q_{v+2}}\ldots\mathrm{E}_{q_m} \\ \mathrm{H}_{\xi_{\upsilon+1}}\mathrm{H}_{\xi_{\upsilon+2}}\ldots\mathrm{H}_{\xi_n}\end{bmatrix}}{\begin{bmatrix} \mathrm{E}_1\mathrm{E}_{q_3} & \ldots\mathrm{E}_{q_v} \\ \mathrm{E}_2\mathrm{E}_{q_{v+1}} & \ldots\mathrm{E}_{q_m}\end{bmatrix}\times\begin{bmatrix} \mathrm{H}_{\xi_1} & \mathrm{H}_{\xi_2} & \ldots\mathrm{H}_{\xi_\upsilon} \\ \mathrm{H}_{\xi_{\upsilon+1}} & \mathrm{H}_{\xi_{\upsilon+2}} & \ldots\mathrm{H}_{\xi_n}\end{bmatrix}}$$

and

$$\frac{(E_2.E_{\nu_2}\ldots E_{\nu_v})\times(H_{\xi_1}H_{\xi_2}\ldots H_{\xi_\nu})\times\begin{bmatrix}E_2 & E_{\nu_2}\ldots E_{\nu_v}\\ H_{\xi_1}H_{\xi_2}\ldots H_{\xi_\nu}\end{bmatrix}\times\begin{bmatrix}E_1 & E_{\nu_{v+1}}\ldots E_{\nu_m}\\ H_{\xi_{\nu+1}}H_{\xi_{\nu+2}}\ldots H_{\xi_n}\end{bmatrix}}{\begin{bmatrix}E_2E_{\nu_2}\ldots E_{\nu_v}\\ E_1E_{\nu_{v+1}}\ldots E_{\nu_m}\end{bmatrix}\times\begin{bmatrix}H_{\xi_1} & H_{\xi_2}\ldots H_{\xi_\nu}\\ H_{\xi_{\nu+1}}H_{\xi_{\nu+2}}\ldots H_{\xi_n}\end{bmatrix}};$$

the sum of this pair of terms will be of the form

$$\frac{P}{Q}\cdot\left\{\frac{E_1}{E_1-E_2}\cdot\frac{\begin{bmatrix}E_1\\ H_{\xi_1}H_{\xi_2}\ldots H_{\xi_\nu}\end{bmatrix}\times\begin{bmatrix}E_2\\ H_{\xi_{\nu+1}}H_{\xi_{\nu+2}}\ldots H_{\xi_n}\end{bmatrix}}{\begin{bmatrix}E_1\\ E_{\nu_{v+1}}E_{\nu_{v+2}}\ldots E_{\nu_m}\end{bmatrix}}\right\}$$

$$+\frac{P}{Q}\cdot\left\{\frac{E_2}{E_2-E_1}\cdot\frac{\begin{bmatrix}E_2\\ H_{\xi_1}H_{\xi_2}\ldots H_{\xi_\nu}\end{bmatrix}\times\begin{bmatrix}E_1\\ H_{\xi_{\nu+1}}H_{\xi_{\nu+2}}\ldots H_{\xi_n}\end{bmatrix}}{\begin{bmatrix}E_2\\ E_{\nu_{v+1}}E_{\nu_{v+2}}\ldots E_{\nu_m}\end{bmatrix}}\right\},$$

where Q, it may be observed, does not contain H_1-H_2, so that $\frac{P}{Q}$ remains finite when $H_1=H_2$.

The above pair of terms together make up a sum of the form

$$\frac{P}{Q}\cdot\frac{1}{E_1-E_2}\cdot\frac{\varphi(E_1,E_2)\psi E_2-\varphi(E_2,E_1)\psi E_1}{\psi E_1\times\psi E_2},$$

which (as the numerator of the third factor vanishes when $E_1=E_2$) remains finite on that supposition. Hence the whole sum of terms in (22.) which is made up of such pairs of terms, and of other terms in which E_1-E_2 does not enter, remains finite when $E_1-E_2=0$, and therefore generally when $D=0$, and similarly when $H_1-H_2=0$, and therefore also when $\Delta=0$; hence the expression for $\mathfrak{S}$ in (22.) is an integral function of the E and H series of quantities, as was to be proved.

Art. (17.). Let us now proceed to determine the dimensions of the coefficient of x^ι, the highest power of x in this value of $\mathfrak{S}_\iota$, when supposed to be expressed under the form of an integral function (as it has been proved to be capable of being expressed) of $h_1\ h_2\ldots h_m;\ \ \eta_1\ \eta_2\ldots\eta_n;\ \ x$.

This coefficient is the sum of fractions the numerators of each of which consist of two factors, which are respectively of $v\times\nu$ and of $(m-v)\times(n-\nu)$ dimensions in respect of the two sets of roots taken conjointly, and the denominators of two factors respectively of $v.(m-v)$ and $\nu\times(n-\nu)$ dimensions in respect of the same.

Consequently, the exponent of the total dimensions of the coefficient in question

$$=v\times\nu+(m-v)(n-\nu)-v(m-v)-\big(\nu.(n-\nu)\big)$$
$$=(m-v-\nu)\times(n-v-\nu)$$
$$=(m-\iota).(n-\iota),$$

and thus is seen to depend only on the degree ι in x of $\mathfrak{S}_{\nu}$, and not upon the mode of partitioning ι into two parts v and ν, for the purpose of representing $\mathfrak{S}_{\nu}$ by means of formula (19.)

Art. (18.). I shall now demonstrate that every form in this scale (to a numerical factor *près*) is identical with a simplified residue to f, φ of the same degree ι in x. Any such simplified residue is like $\mathfrak{S}_{\nu}$ a syzygetic function, or to use a briefer form of speech, a conjunctive of f, φ; and if we agree to understand by the "weight" of any function of the coefficients of f and φ its joint dimensions in respect of the roots of f and φ combined, I shall prove,—1st, that any simplified residue of f and φ of a given degree in x is that conjunctive, whose weight in respect of the roots of f and φ is less than the weight of any other such conjunctive; and 2nd, that $\mathfrak{S}_{\nu}$ as determined above (in equation 24.), is of the same weight as the simplified residue, and can therefore only differ from it by some numerical factor. For the purpose of comparison of weights, it will of course be sufficient to confine our attention to the coefficients of the highest (or any other, the same power, for each) in x of the forms whose weights are to be compared.

Suppose f to be of m dimensions, and φ to be of n dimensions in x; and let $m=n+e$.

Suppose

$$\Lambda.f+\mathrm{L}.\varphi=\mathrm{A}x^{\iota}+\mathrm{B}x^{\iota-1}+\&\mathrm{c.}+\mathrm{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23.)$$

$$\Lambda=\lambda_0.x^{q}+\lambda_1.x^{q-1}+\&\mathrm{c.}+\lambda_q$$

$$\mathrm{L}=l_0x^{q+e}+l_1.x^{q+e-1}+\&\mathrm{c.}+l_{q+e}$$

the number of homogeneous equations to be satisfied by the $q+1$ quantities $\lambda_0\lambda_1\ldots\lambda_q$, and the $q+\delta+1$ quantities $\mu_0, \mu_1,\ldots\mu_{q+e}$ will be $m+q-i$, and therefore $q+1$ and $q+e+1$ taken together must be not less than $m+q-i+1$, *i. e.* $2q+e+2$ must be not less than $q+m-i+1$, *i. e.* q not less than $m-i-e-1$; and if this inequality be satisfied $2q+e+2-(q+m-i-1)+1$, *i. e.* $q+i+e-m+2$ will be the number of arbitrary constants entering into the solution of equation (23.).

If q be greater than $(n-1)$, let $q=(n-1)+t$;

and let $$(\Lambda)=(\lambda_0).x^{n-1}+(\lambda_1)x^{n-2}+\ldots+(\lambda_{n-1})$$

$$(\mathrm{L})=l_0.x^{n+e-1}+l_1.x^{n+e-2}+\ldots+l(x_{e+n-1});$$

and let (Λ), (L) be so taken as to satisfy the equation

$$(\Lambda)f+(\mathrm{L}).\varphi=\mathrm{A}x^{\iota}+\mathrm{B}x^{-1}+\ldots+\mathrm{K};$$

and make $$\Xi=(\Lambda)+(f+gx+\ldots+hx^{t-1}).\varphi$$

$$\mathrm{X}=(\mathrm{L})-(f+gx+\ldots+hx^{t-1})f,$$

$f, g,\ldots h$ being arbitrary constants;

then $$\Xi f+\mathrm{X}.\varphi=(\Lambda)f+(\mathrm{L})\varphi=\mathrm{A}x^{\iota}+\mathrm{B}x^{-1}+\ldots+\mathrm{K}.$$

Now the total number of arbitrary constants in the system (Λ) and (L) will be $n-1+i+e-m+2$, *i. e.* $i+1$; hence the total number of arbitrary constants in Ξ and

X will be $i+1+t$, *i. e.* $q-n+i+2$, which is equal to $q+i+e-m+2$, the number of arbitrary constants in the most general values of Λ and L. Hence $\{\Lambda=\Xi;\ \mathrm{L}=\mathrm{X}\}$ is the general solution of $\Lambda.f+\mathrm{L}.\varphi=\mathrm{A}x^i+\mathrm{B}x^{i-1}+\ldots+\mathrm{K}$; and consequently the most general form of $\mathrm{A}x^i+\mathrm{B}x^{i-1}+\ldots+\mathrm{K}$, which is evidently independent of the (t) arbitrary quantities $f, g \ldots h$, will contain the same number of arbitrary constants as enter into the system (Λ) and (L), *i. e.* $i+1$.

Art. (19.). Let us now begin with the case of greater simplicity when $m=n$, *i. e.* $e=0$; and let us revert to the system of equations marked (10.) in Section I., in which U and V are to be replaced by f and φ.

1st. Let $i=n-1$, and therefore $i+1$, the number of arbitrary quantities in the conjunctive is n.

From the system of equations (10.), we have for all values of $\varrho_1, \varrho_2, \varrho_3 \ldots \varrho_n$,

$$(\varrho_1\mathrm{Q}_0+\varrho_2\mathrm{Q}_1+\ldots+\varrho_n.\mathrm{Q}_{n-1})f$$
$$-(\varrho_1\mathrm{P}_0+\varrho_2.\mathrm{P}_1+\ldots+\varrho_n.\mathrm{P}_{n-1})\varphi$$
$$=(\varrho_1.\mathrm{K}_1+\varrho_2.{}_1\mathrm{K}_1+\ldots+\varrho_n.{}_{n-1}\mathrm{K}_1)x^{n-1}+\&\mathrm{c.},$$

and consequently the most *general* value of $\mathfrak{S}_{n-1}$ in the equation

$$\tau_{n-1}.f-t_{n-1}.\varphi+\mathfrak{S}_{n-1}=0,$$

where $\mathfrak{S}_{n-1}=\mathrm{A}x^{n-1}+\mathrm{B}x^{n-2}+\ldots+\mathrm{L}$

will be obtained by making

$$\tau_{n-1}=\varrho_1.\mathrm{Q}_0+\varrho_2.\mathrm{Q}_1+\ldots+\varrho_n.\mathrm{Q}_n$$
$$t_{n-1}=-\varrho_1.\mathrm{P}_0-\varrho_2.\mathrm{P}_1\ldots-\varrho_n.\mathrm{P}_n,$$

which solution contains n, *i. e.* the proper number of arbitrary contants.

Again, if $i=n-2$ $i+1=n-1$, which will therefore be the number of arbitrary constants in the most general value of $\mathfrak{S}_{n-2}$ of the equation

$$\tau_{n-2}.f-t_{n-2}.\varphi+\mathfrak{S}_{n-2}=0.$$

This most general value of $\mathfrak{S}_{n-2}$ is therefore found by making

$$\tau_{n-2}=\varrho'_1\mathrm{Q}_0+\varrho'_2.\mathrm{Q}_1+\ldots+\varrho'_n.\mathrm{Q}_n$$
$$t_{n-1}=-\varrho'_1\mathrm{P}_0-\varrho'_2.\mathrm{P}_1\ldots-\varrho'_n.\mathrm{P}_n,$$

where $\varrho'_1, \varrho'_2, \ldots \varrho'_n$ are no longer entirely independent, but subject to the equation

$$\varrho'_1.\mathrm{K}_1+\varrho'_2.{}_1\mathrm{K}_1+\ldots+\varrho'_n.{}_{n-1}\mathrm{K}_1=0,$$

so as to leave $(n-1)$ constants arbitrary.

We thus obtain $\mathfrak{S}_{n-2}=(\varrho'_1\mathrm{K}_2+\varrho'_2.{}_1\mathrm{K}_2+\ldots+\varrho'_n.{}_{n-1}\mathrm{K}_2)x^{n-2}+\&\mathrm{c.}$ In like manner, and for the same reasons, the most general values of $\mathfrak{S}_{n-3}$ in the equation

$$\tau_{n-3}.f-t_{n-3}.\varphi+\mathfrak{S}_{n-3}=0$$

will be found by making

$$\tau_{n-3}=\varrho''_1.\mathrm{Q}_0+\varrho''_2.\mathrm{Q}_1+\ldots+\varrho''_n.\mathrm{Q}_{n-1}$$
$$t_{n-3}=-\varrho''_1.\mathrm{P}_0-\varrho''_2.\mathrm{Q}_1\ldots-\varrho''_n.\mathrm{P}_{n-1}.$$

where $\varrho_1'', \varrho_2'', \ldots \varrho_n''$ are subject to satisfying the two equations

$$\varrho_1''.K_1+\varrho_2''.{}_1K_1+\ldots+\varrho_{n-1}''.{}_{n-2}K_1=0$$
$$\varrho_1''.K_2+\varrho_2''.{}_1K_2+\ldots+\varrho_{n-1}''.{}_{n-2}K_2=0,$$

so as to leave $(n-2)$ constants arbitrary; and we thus obtain

$$\Theta_{n-3}=(\varrho_1''.K_3+\varrho_2''.{}_4K_2+\ldots+\varrho_{n-1}''.{}_{n-1}K_3)x^{n-3}+\&c.,$$

and so on, the number of independent arbitrary constants in Θ decreasing (as it ought) each time by one unit as the degree of Θ descends, until finally, if $\tau_0.f-t_0.\varphi+\Theta_0=0$, Θ_0 being a constant, the general value for Θ_0 is found by making

$$\tau_0=(\varrho_1).Q_0+(\varrho_2)Q_1+\ldots+(\varrho_n).Q_{n-1}$$
$$t_0=-(\varrho_1)P_0-(\varrho_2)P_1-\ldots-(\varrho_n).P_{n-1},$$

where $\varrho_1, \varrho_2, \ldots \varrho_n$ are subject to satisfy the $(n-1)$ equations

$$(\varrho_1).K_1+\&c.=0$$
$$(\varrho_1).K_2+\&c.=0$$
$$\ldots\ldots$$
$$(\varrho_1).K_{n-1}+\&c.=0,$$

which gives $$\Theta_0=K_n(\varrho)_1+{}_2K_n(\varrho)_n+\ldots+{}_{n-1}K_n(\varrho)_n.$$

Now evidently the lowest weight in respect to the roots of U and V that can be given to $(\varrho_1K_1+\varrho_2{}_1K_1+\ldots+\varrho_n{}_{n-1}K_1)x^{n-1}+\&c.$, when the multipliers $\varrho_1, \varrho_2, \ldots \varrho_n$ are absolutely independent, is found by taking $\varrho_1=1$ $\varrho_2=0$ $\varrho_3=0\ldots\varrho_n=0$, which makes the weight of the leading coefficient in Θ_{n-1}, the same as that of K_1, *i. e.* 1.

Again, when one equation,

$$\varrho_1K_1+\varrho_2'{}_1K_1+\ldots+\varrho_n'{}_{n-1}K_1=0,$$

exists between the (ϱ)'s, the lowest weight will be found by making

$$\varrho_1'={}_1K_1 \quad \varrho_2'=-K_1 \quad \varrho_3'=0 \quad \varrho_4'=0\ldots\varrho_n'=0,$$

which makes the weight of the leading coefficient in Θ_{n-2} depend on

$${}_1K_1K_2-K_1{}_1K_2,$$

which is of the weight $1+3$, *i. e.* 4 in respect of the roots of f and φ.

Similarly, Θ_{n-3} will have its lowest weight when its leading coefficient is the determinant

$$\begin{matrix} K_1 & K_2 & K_3 \\ {}_1K_1 & {}_1K_2 & {}_1K_3 \\ {}_2K_1 & {}_2K_2 & {}_2K_3, \end{matrix}$$

the weight of which is $1+3+5=9$; and finally, the lowest weighted value of Θ_0 is the determinant represented by the complete Bezoutian square; the weight in general of Θ_{n-i} being $1+3+\ldots+(2i-1)$, *i. e.* i^2, or which is the same thing otherwise expressed, the weight of the leading coefficient of the lowest-weighted conjunctive of f and φ of the degree i in x is $(n-i)(m-i)$*. It will of course have been seen in the fore-

* n and m are supposed equal and $i=n-i$.

going demonstration, that the weight of ${}_rK'_s$ [which means $\Sigma(a_r.b_s - a_s.b_r)\,a_r,\,a_s$ being the coefficients of x^{n-r}, x^{n-s} in f, and b_r, b_s of the same in ϕ] has been correctly taken to be $r+s$ in respect of the roots of f and ϕ conjoined.

Art. (20.). If now we proceed in like manner with the general case of $m=n+e$, it may be shown, in precisely the same way as in the preceding article, that the most general value of any conjunctive of f and ϕ will be a linear function of (e) functions,

$$\begin{aligned}
&x^n + a_1.x^{n-1} + a_2\,x^{n-2} + \ldots + a_n\\
&x^{n+1} + a_1.x^n + a_2.x^{n-1} + \ldots + a_n.x\\
&x^{n+2} + a_1.x^{n+1} + a_2.x^n + \ldots + a_n\,x^2\\
&\quad\ldots\ldots\ldots\ldots\\
&\quad\ldots\ldots\ldots\ldots\\
&x^{m-1} + a_1.x^{m-2} + \&c. \qquad\qquad + a_n x^{e-1},
\end{aligned}$$

and of the (n) functions,

$$\begin{aligned}
&{}_0K_1.x^{n-1} + {}_0K_2.x^{n-2} + \ldots + {}_0K_n\\
&{}_1K_1.x^{n-1} + {}_1K_2.x^{n-2} + \ldots + {}_1K_n\\
&\qquad \&c. \qquad\qquad \&c.\\
&{}_{n-1}K_1.x^{n-1} + {}_{n-1}K_2\,x^{n-2} + \ldots + {}_{n-1}K_n,
\end{aligned}$$

and that consequently, if the degree of such conjunctive in x be $(n-i)$, it will be of the lowest weight when it is a linear function of the entire (e) upper set of functions, and (i) of the lower set; and consequently, the coefficient of the highest power of x in such conjunctive will be the determinant

$$\begin{array}{cccccc}
{}_0K_1 & {}_0K_2 & {}_0K_3 \ldots\ldots {}_0K_i \ldots\ldots\ldots\ldots & {}_0K_{i+e}\\
{}_1K_1 & {}_1K_2 & {}_1K_3 \ldots\ldots {}_1K_i \ldots\ldots\ldots\ldots & {}_1K_{i+e}\\
{}_2K_1 & {}_2K_2 & {}_2K_3 \ldots\ldots {}_2K_i \ldots\ldots\ldots\ldots & {}_2K_{i+e}\\
\ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots\\
\ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots\\
{}_{i-1}K_1 & {}_{i-1}K_2 & {}_{i-1}K_3 \ldots {}_{i-1}K_i \ldots\ldots\ldots\ldots & {}_{i-1}K_{i+e}\\
1 & a_1 & a_2 \ldots\ldots a_{i-1} \quad a_i \ldots\ldots\ldots\ldots & a_{i+e}\\
 & 1 & a_1 \ldots\ldots a_{i-1} \quad a_i \ldots\ldots\ldots\ldots & a_{i+e-1}\\
 & & 1 \ldots\ldots a_{i-2} \quad a_i \ldots\ldots\ldots\ldots & a_{i+e-2}\\
 & & \ldots\ldots\ldots\ldots & \ldots\\
 & & \ldots\ldots\ldots\ldots & \ldots\\
 & & 1 \quad \ldots\ldots & a_e
\end{array}$$

the weight of which is evidently that of

$${}_0K_1 \times {}_1K_2 \times {}_2K_3 \ldots \times {}_{i-1}K_i \times (a_i)^e,$$

i. e. $1+3+5+\ldots+(2i-1)+e.i$

i. e. i^2+ei, or $i(e+i)$, which is $(n-i)(m-i)$ if $i=n-i$.

Hence the weight of the leading coefficient in the lowest-weighted conjunctive of f and φ of the degree ι in x is $(m-\iota)(n-\iota)$, m being the degree of f and n of φ.

From this we infer that any conjunctive of f and φ of the degree ι, of which the leading coefficient is of the weight $(m-\iota)(n-\iota)$ (all the coefficients being of course understood to be integral functions of the roots of f and φ), must, to a numerical factor *près*, be equivalent to any other of the same weight; and furthermore, any supposed function of x of the ιth degree which possesses the property characteristic of a conjunctive of vanishing, when f and φ vanish simultaneously, but of which the weight of the leading coefficient would be *less* than $(m-\iota)(n-\iota)$, must be a mere nugatory form and have all its terms *identically zero**.

Art. (21.). We have previously shown, art. (16.), that $\mathfrak{D}$, as defined by equation (21.), is an integral function of the roots f and φ, and vanishes when f and φ vanish. Moreover, its weight in the roots has been proved to be $(m-\iota)(n-\iota)$, and consequently, if by way of distinguishing the several forms of $\mathfrak{D}_\iota$ we name that one where ι in the equation above cited is supposed to be divided into two parts, v and ν, $\mathfrak{D}_{v,\nu}$, we have for all values of v and ν, such that $v+\nu$ is not greater than n, $\mathfrak{D}_{v,\nu}$ to a constant numerical factor *près* identical with the $(v+\nu)$th simplified residue to (f, φ), so that the form of $\mathfrak{D}_{v,\nu}$ depends only upon the value of $v+\nu$.

Art. (22.). It must be well borne in mind that this permanency of the value of $\mathfrak{D}_{v,\iota-v}$ for different values of v has only been established for the case where i can be the degree of a residue to f and φ, that is to say, when i is less than the lesser of the two indices m and n. When i does not satisfy this condition of inequality, the theorem ceases to be true. It is clear that when $m=n$ and $v+\nu=m=n$, $\mathfrak{D}_{v,\nu}$ which always remains a conjunctive of f and φ, can only be a numerical linear function of f and φ; and I have ascertained when $m=n$ on giving to v and ν the respective values successively $(0, n)$, $(1, n-1)$, $(2, (n-2))$, ... $(n, 0)$

that $\mathfrak{D}_{0,n}=f$; $\mathfrak{D}_{1,n-1}=(n-1)f+\varphi$; $\mathfrak{D}_{2,n-2}=\frac{(n-1)(n-2)}{1.2}f+(n-1)\varphi$...

$$\mathfrak{D}_{n-1,1}=f+(n-1)\varphi, \mathfrak{D}_{n,0}=\varphi.$$

Thus, by way of a simple example, let

$$f=x^3+ax+b=(x-h_1)(x-h_2)$$

$$\varphi=x^3+\alpha x+\beta=(x-\eta_1)(x-\eta_2)$$

$$\therefore \mathfrak{D}_{0,2}=(x-h_1)(x-h_2)\left\{\frac{\begin{bmatrix} h_1h_2 \\ .. \end{bmatrix}\times\begin{bmatrix} .. \\ k_1k_2 \end{bmatrix}}{\begin{bmatrix} h_1h_2 \\ .. \end{bmatrix}\times\begin{bmatrix} .. \\ k_1k_2 \end{bmatrix}}\right\}=(x-h_1)(x-h_2)=f$$

* And more generally it admits of being demonstrated by precisely the same course of reasoning, that the number of arbitrary parameters in a conjunctive of the degree i, and of the weight $(m-i)(n-i)+s$ in the roots cannot (abstraction being supposed to be made of an arbitrary numerical multiplier) exceed the number s.

$$\mathfrak{S}_{1,1}=\Sigma(x-h_1)(x-k_1)\frac{\begin{bmatrix}h_1\\k_1\end{bmatrix}\times\begin{bmatrix}h_2\\k_2\end{bmatrix}}{\begin{bmatrix}h_1\\h_2\end{bmatrix}\times\begin{bmatrix}k_1\\k_2\end{bmatrix}}$$

$$=\Sigma\tfrac{x-h_1}{h_1-h_2}\Sigma\tfrac{x-k_1}{k_1-k_2}.\{(h_1-k_1)(h_2-k_2)\},$$

$$i.\,e.\ =\Sigma\tfrac{x-h_1}{h_1-h_2}\cdot\left\{\tfrac{1}{k_1-k_2}\begin{pmatrix}(x-k_1)(h_1-k_1)(h_2-k_2)\\-(x-k_2)(h_1-k_2)(h_2-k_1)\end{pmatrix}\right\}$$

$$=\Sigma\tfrac{x-h_1}{h_1-h_2}\cdot\left\{(h_1-h_2)x+\{(k_1+k_2)h_2-(h_1h_2+k_1k_2)\}\right\}$$

$$=(x-h_1)x+(x-h_2)x-(k_1+k_2)x+(h_1h_2+k_1k_2)$$

$$=(x^2-(h_1+h_2)x+h_1h_2)+(x^2-(k_1+k_2)x+k_1k_2)$$

$$=(x^2+ax+b)+(x^2+\alpha x+\beta)$$

$$=f+\varphi;$$

so we find also $\mathfrak{S}_{2,0}=\varphi$.

Art. (23.). The expression $\mathfrak{S}_{v,\nu}$, which is universally a conjunctive of f and φ, continues algebraically interpretable so long as $v+\nu$ has any value intermediate between (0) and $m+n$; when $v+\nu=0$, we must of course have $v=0$ and $\nu=0$, and $\mathfrak{S}_{0,0}$ becomes the resultant of f and φ when $v+\nu=m+n$; we must also have the unique solution $v=m$ and $\nu=n$, and $\mathfrak{S}_{m,n}$ becomes necessarily $f\times\varphi$, which we thus see stands in a sort of antithetical relation to the resultant of f and φ, say (f,φ). Nor is it without interest to remark that $f\times\varphi=0$ implies that a root of f or else of φ is zero; and $(f,\varphi)=0$ implies that if a root of the one of the functions is zero, so also is a root of the other, *i. e.* that a root of each or of neither is zero. As i increases from 0 to n or decreases from $m+n$ to $m-1$, the number of solutions of the equation $v+\nu=i$ in the one case, and the number of admissible solutions of the equation $v+\nu=i$ in the other case, which is subject to the condition that ν must not exceed n, continues to increase by a unit at each step; there being thus $n+1$ different forms $\mathfrak{S}_{v,\nu}$ when $v+\nu=n$, and the same number when $v+\nu=m-1$. For all values of i intermediate between n and $(m-1)$ (both taken exclusively) it is very remarkable that $\mathfrak{S}_{v,\nu}$ will vanish, as I proceed to demonstrate.

Art. (24.). The weight of the coefficient of the highest power of $\mathfrak{S}_{v,\nu}$ ($v+\nu$ being equal to i) is $(m-i)(n-i)$, and consequently, when i is greater than n, and less than m, $\mathfrak{S}_{v,\nu}$ would contain fractional functions of the roots of f and φ, if there were in it a power x^i, but $\mathfrak{S}_{v,\nu}$ has been proved to be always an integer function of the roots. Hence the coefficient of x^i will be zero, and so more generally the first power of x in $\mathfrak{S}_{v,\nu}$, of which the coefficient is not zero, will be x^ω, subject to the condition (since evidently the weight of the several coefficients goes on increasing by units as the degree of the terms in x decreases by the same) that ω be not less than $(m-i)(i-n)$; let then $\omega=(m-i)(i-n)$, $\mathfrak{S}_{v,\nu}$ becomes of the form $Ax^{i-\omega}+Bx^{i-\omega-1}+$ &c., where A is of zero dimensions; but this is impossible if $i-\omega<n$, for then $Ax^{i-\omega}+$ &c. is a conjunctive of

weight lower than the lowest-weighted simplified residue of the degree $i-\omega$. Hence ω is not greater than $i-n$ or $(m-i)(i-n)$ is not greater than $i-n$, *i. e.* $m-i$ cannot be greater than 1, *i. e.* i when intermediate between m and n cannot be less than $m-1$, otherwise $\mathfrak{S}_{v,\nu}$ will vanish identically. Moreover, when $i=m-1$, $\omega=i-n$, and $i-\omega=n$, and accordingly $\mathfrak{S}_{v,m+1-v}$ is not merely, as we might know, *à priori* an algebraical, but more simply a numerical multiple of φ for all values of v. The same is of course true also, m being greater than n, for every form $\mathfrak{S}_{v,n-v}$, since this is always a conjunctive of f and φ, of which the former is of a degree higher than the $\mathfrak{S}$ in question, so that the multiplier of f in this conjunctive must be zero*.

Art. (25.). To enter into a further or more detailed examination of the values assumed by $\mathfrak{S}_{v,\nu}$ for the most general values of m, n, i, would be to transcend the limits I have proposed to myself in drawing up the present memoir. What we have established is, that to every form of $\mathfrak{S}_{v,i-v}$ appertaining to a value of i between 0 and n, there is a sort of conjugate form for which i lies between $m+n$ and m; that for $i=m-1$ or $i=n$, $\mathfrak{S}_{v,i-v}$ becomes a numerical multiplier of φ; and that when i lies in the intermediate region between n and $m-1$, $\mathfrak{S}_{v,i-v}$ vanishes for all values of v. I pause only for a moment to put together for the purpose of comparison the forms corresponding to i and to $m+n-i$. By art. (16.), making $i=v+\nu$,

$$\mathfrak{S}_\nu=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_v})\times(x-\eta_{\xi_1})(x-\eta_{\xi_2})\ldots(x-\eta_{\xi_\nu})$$
$$\times\frac{\begin{bmatrix}h_{q_1}&h_{q_2}&\ldots&h_{q_v}\\ \eta_{\xi_1}&\eta_{\xi_2}&\ldots&\eta_{\xi_\nu}\end{bmatrix}\times\begin{bmatrix}h_{q_{v+1}}&h_{q_{v+2}}&\ldots&h_{q_m}\\ \eta_{\xi_{\nu+1}}&\eta_{\xi_{\nu+2}}&\ldots&\eta_{\xi_n}\end{bmatrix}}{\begin{bmatrix}h_{q_1}&h_{q_2}&\ldots&h_{q_v}\\ h_{q_{v+1}}&h_{q_{v+2}}&\ldots&h_{q_m}\end{bmatrix}\times\begin{bmatrix}\eta_{\xi_1}&\eta_{\xi_2}&\ldots&\eta_{\xi_\nu}\\ \eta_{\xi_{\nu+1}}&\eta_{\xi_{\nu+2}}&\ldots&\eta_{\xi_n}\end{bmatrix}}.$$

The conjugate form for which $i=m+n-i$ and $m-v$ $n-\nu$ take the places of v and ν $(m-v)(n-\nu)$ will be got by taking

$$\mathfrak{S}_{v'}=\Sigma(x-h_{q_{v+1}})(x-h_{q_{v+2}})\ldots(x-h_{q_m})\times(x-\eta_{\xi_{\nu+1}})(x-\eta_{\xi_{\nu+2}})\ldots(x-\eta_{\xi_n})$$
$$\times\frac{\begin{bmatrix}h_{q_1}&h_{q_2}&\ldots&h_{q_v}\\ \eta_{\xi_1}&\eta_{\xi_2}&\ldots&\eta_{\xi_\nu}\end{bmatrix}\times\begin{bmatrix}h_{q_{v+1}}&h_{q_{v+2}}&\ldots&h_{q_m}\\ \eta_{\xi_{\nu+1}}&\eta_{\xi_{\nu+2}}&\ldots&\eta_{\xi_n}\end{bmatrix}}{\begin{bmatrix}h_{q_1}&h_{q_2}&\ldots&h_{q_v}\\ h_{q_{v+1}}&h_{q_{v+2}}&\ldots&h_{q_m}\end{bmatrix}\times\begin{bmatrix}\eta_{\xi_1}&\eta_{\xi_2}&\ldots&\eta_{\xi_\nu}\\ \eta_{\xi_{\nu+1}}&\eta_{\xi_{\nu+2}}&\ldots&\eta_{\xi_n}\end{bmatrix}},$$

which it will be perceived are identical, term for term, in the fractional constant factor, and differ only in the linear functions of x, which in $\mathfrak{S}_i$ and in $\mathfrak{S}_{i'}$ are complementary to one another. Our proper business is only with those forms for which $i<n$.

Art. (26.). It will presently be seen to be necessary to ascertain the numerical relations between $\mathfrak{S}_{0,i}$ and $\mathfrak{S}_{i,0}$ when $i<n$, and this naturally brings under our notice the

* It thus appears that if the indices m and n do not differ by at least 3 units, $\mathfrak{S}_i$ will have an actual quantitative existence for all values of i between 0 and $m+n$; or in other words, the failure in the quantitative existence of the forms $\mathfrak{S}_i$ only begins to show itself when this difference is 3; thus if $m=n+3$, $\mathfrak{S}_n$ exists, and $\mathfrak{S}_{n+2}$ exists, but $\mathfrak{S}_{n+1}=0$.

inquiry into the numerical relations which exist between the entire series of forms $\vartheta_{v,i-v}$ for a given value of i, corresponding to all values of v between 0 and i inclusive.

In order to avoid a somewhat oppressive complication of symbols, I shall take a particular numerical example, *i. e.* $m=7$ $n=6$ $i=4$, and compare the values of $\vartheta_{0,4}$; $\vartheta_{1,3}$; $\vartheta_{2,2}$; $\vartheta_{3,1}$; $\vartheta_{4,0}$, all of which we know to be identical, [to a numerical factor *près*] with one another and with the second simplified residue to f and ϕ, that being of the fourth degree in x; our object in the subjoined investigation is to determine the numerical ratios of these several forms of ϑ to one another.

First. Let $v=0$ $\nu=4$. The leading coefficient $\vartheta_{0,4}$ is

$$\Sigma \frac{h_1 h_2 h_3 h_4 h_5 h_6 h_7}{\eta_1 \eta_2 \eta_3 \eta_4}$$

which we know *à priori* (it should be observed) to be essentially an *integral* function of the h and the η system. In this, the term containing η_6^3 will be evidently

$$\text{(A.)} \qquad \Sigma \frac{h_1 h_2 h_3 h_4 h_5 h_6 h_7}{\eta_1 \eta_2 \eta_3 \eta_4},$$

the η system to which the latter summation relates being now reduced to consist of $\eta_1 \eta_2 \eta_3 \eta_4 \eta_5$. In this expression, again, the coefficient of η_5^3 is evidently 1. Hence, therefore, the leading coefficient in $\vartheta_{0,4}$ contains the term $\eta_5^3 . \eta_6^3$.

Secondly. Let $v=1$ $\nu=3$. The leading coefficient in $\vartheta_{1,3}$ becomes

$$\Sigma \frac{\begin{bmatrix} \eta_1 \eta_2 \eta_3 \\ h_1 \end{bmatrix} \times \begin{bmatrix} \eta_4 \eta_5 \eta_6 \\ h_2 h_3 h_4 h_5 h_6 h_7 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 h_4 h_5 h_6 h_7 \\ h_1 \end{bmatrix} \times \begin{bmatrix} \eta_4 \eta_5 \eta_6 \\ \eta_1 \eta_2 \eta_3 \end{bmatrix}}.$$

In this, the factor affecting η_6^2 will be

$$\Sigma \frac{\begin{bmatrix} \eta_1 \eta_2 \eta_3 \\ h_1 \end{bmatrix} \times \begin{bmatrix} \eta_4 \eta_5 \\ h_2 h_3 h_4 h_5 h_6 h_7 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 h_4 h_5 h_6 h_7 \\ h_1 \end{bmatrix} \times \begin{bmatrix} \eta_4 \eta_5 \\ \eta_1 \eta_2 \eta_3 \end{bmatrix}},$$

η_6 being now understood to be eliminated out of the η system included within the above summation. Again, in this latter sum the factor affecting η_5^2 will be

$$\text{(B.)} \qquad \Sigma \frac{\begin{bmatrix} \eta_1 \eta_2 \eta_3 \\ h_1 \end{bmatrix} \times \begin{bmatrix} \eta_4 \\ h_2 h_3 h_4 h_5 h_6 h_7 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 h_4 h_5 h_6 h_7 \\ h_1 \end{bmatrix} \times \begin{bmatrix} \eta_4 \\ \eta_1 \eta_2 \eta_3 \end{bmatrix}},$$

η_5 and η_6 being now both eliminated out of the η system. This last sum can of course only represent a numerical quantity.

So in like manner, again, if $v=2$ $\nu=2$, the coefficient of $\eta_6^2.\eta_7^2$ will be similarly reducible to the form

$$\text{(C.)} \qquad \Sigma \begin{bmatrix} \eta_1 \eta_2 \\ h_1 h_2 \end{bmatrix} \times \begin{bmatrix} \eta_3 \eta_4 \\ h_3 h_4 h_5 h_6 h_7 \end{bmatrix} \Big/ \begin{bmatrix} h_3 h_4 h_5 h_6 h_7 \\ h_1 h_2 \end{bmatrix} \times \begin{bmatrix} \eta_3 \eta_4 \\ h_1 h_2 \end{bmatrix} \text{ in } \mathfrak{S}_{2,2}.$$

So, again, when $v=3$ $\nu=1$, the coefficient of $\eta_6^2.\eta_7^2$ will be

$$\text{(D.)} \qquad \Sigma \begin{bmatrix} \eta_1 \\ h_1 h_2 h_3 \end{bmatrix} \times \begin{bmatrix} \eta_2 \eta_3 \eta_4 \\ h_4 h_5 h_6 h_7 \end{bmatrix} \Big/ \begin{bmatrix} h_4 h_5 h_6 h_7 \\ h_1 h_2 h_3 \end{bmatrix} \times \begin{bmatrix} \eta_2 \eta_3 \eta_4 \\ \eta_1 \end{bmatrix} \text{ in } \mathfrak{S}_{2,1};$$

and finally, the coefficient of $\eta_6^2.\eta_7^2$ will be

$$\text{(E.)} \qquad \Sigma \begin{bmatrix} \eta_1 \eta_2 \eta_3 \eta_4 \\ h_5 h_6 h_7 \end{bmatrix} \Big/ \begin{bmatrix} h_5 h_6 h_7 \\ h_1 h_2 h_3 h_4 \end{bmatrix} \text{ in } \mathfrak{S}_{0,4},$$

out of all which sums it is to be remembered that η_5 and$_6$ are supposed excluded from appearing. All these several coefficients being numbers in disguise, we may determine them by giving any values at pleasure to the terms in the h and η system.

Let now $\eta_1=h_1$ $\eta_2=h_2$ $\eta_3=h_3$ $\eta_4=h_4$, then in (B.) it will readily be seen that all the terms included within the sign of summation vanish identically, except the following, viz.—

$$\begin{bmatrix} \eta_1 \eta_2 \eta_3 \\ h_4 \end{bmatrix} \times \begin{bmatrix} \eta_4 \\ h_1 h_2 h_3 h_5 h_6 h_7 \end{bmatrix} \Big/ \begin{bmatrix} h_1 h_2 h_3 h_5 h_6 h_7 \\ h_4 \end{bmatrix} \times \begin{bmatrix} \eta_4 \\ \eta_1 \eta_2 \eta_3 \end{bmatrix}$$

$$\begin{bmatrix} \eta_1 \eta_2 \eta_4 \\ h_3 \end{bmatrix} \times \begin{bmatrix} \eta_3 \\ h_1 h_2 h_4 h_5 h_6 h_7 \end{bmatrix} \Big/ \begin{bmatrix} h_1 h_2 h_4 h_5 h_6 h_7 \\ h_3 \end{bmatrix} \times \begin{bmatrix} \eta_3 \\ \eta_1 \eta_2 \eta_4 \end{bmatrix}$$

$$\begin{bmatrix} \eta_1 \eta_3 \eta_4 \\ h_2 \end{bmatrix} \times \begin{bmatrix} \eta_2 \\ h_1 h_3 h_4 h_5 h_6 h_7 \end{bmatrix} \Big/ \begin{bmatrix} h_1 h_3 h_4 h_5 h_6 h_7 \\ h_2 \end{bmatrix} \times \begin{bmatrix} \eta_2 \\ \eta_1 \eta_3 \eta_4 \end{bmatrix}$$

$$\begin{bmatrix} \eta_2 \eta_3 \eta_4 \\ h_1 \end{bmatrix} \times \begin{bmatrix} \eta_1 \\ h_2 h_3 h_4 h_5 h_6 h_7 \end{bmatrix} \Big/ \begin{bmatrix} h_2 h_3 h_4 h_5 h_6 h_7 \\ h_1 \end{bmatrix} \times \begin{bmatrix} \eta_1 \\ \eta_2 \eta_3 \eta_4 \end{bmatrix}.$$

In each of these expressions the first factor of the numerator is identical in value

(by reason of the equations $h_1=\eta_1\ h_2=\eta_2\ h_3=\eta_3\ h_4=\eta_4$) with $(-)^3\times$ the second factor of the denominator, and the second factor of the numerator with $(-)^6\times$ the first factor of the denominator; hence the coefficient of $\eta_5^2.\eta_6^2$ in $\Theta_{1,2}$ is -4.

In like manner the only effective terms of $\Theta_{2,2}$ will be

$$\frac{\begin{bmatrix}\eta_1\ \eta_2\\ h_3\ h_4\end{bmatrix}\times\begin{bmatrix}\eta_3\ \eta_4\\ h_1\ h_2\ h_5\ h_6\ h_7\end{bmatrix}}{\begin{bmatrix}h_1\ h_2\ h_5\ h_6\ h_7\\ h_3\ h_4\end{bmatrix}\times\begin{bmatrix}\eta_5\ \eta_6\\ \eta_1\ \eta_2\end{bmatrix}},\quad \frac{\begin{bmatrix}\eta_3\ \eta_4\\ h_1\ h_2\end{bmatrix}\times\begin{bmatrix}\eta_1\ \eta_2\\ h_3\ h_4\ h_5\ h_6\ h_7\end{bmatrix}}{\begin{bmatrix}h_3\ h_4\ h_5\ h_6\ h_7\\ h_1\ h_2\end{bmatrix}\times\begin{bmatrix}\eta_1\ \eta_2\\ \eta_3\ \eta_4\end{bmatrix}}$$

$$\frac{\begin{bmatrix}\eta_1\ \eta_3\\ h_2\ h_4\end{bmatrix}\times\begin{bmatrix}\eta_2\ \eta_4\\ h_1\ h_3\ h_5\ h_6\ h_7\end{bmatrix}}{\begin{bmatrix}h_1\ h_3\ h_5\ h_6\ h_7\\ h_2\ h_4\end{bmatrix}\times\begin{bmatrix}\eta_2\ \eta_4\\ \eta_1\ \eta_3\end{bmatrix}},\quad \frac{\begin{bmatrix}\eta_2\ \eta_4\\ h_1\ h_3\end{bmatrix}\times\begin{bmatrix}\eta_1\ \eta_3\\ h_2\ h_4\ h_5\ h_6\ h_7\end{bmatrix}}{\begin{bmatrix}h_2\ h_4\ h_5\ h_6\ h_7\\ h_1\ h_3\end{bmatrix}\times\begin{bmatrix}\eta_1\ \eta_3\\ \eta_2\ \eta_4\end{bmatrix}}$$

$$\frac{\begin{bmatrix}\eta_1\ \eta_4\\ h_2\ h_3\end{bmatrix}\times\begin{bmatrix}\eta_2\ \eta_3\\ h_1\ h_4\ h_5\ h_6\ h_7\end{bmatrix}}{\begin{bmatrix}h_1\ h_4\ h_5\ h_6\ h_7\\ h_2\ h_3\end{bmatrix}\times\begin{bmatrix}\eta_1\ \eta_4\\ \eta_2\ \eta_3\end{bmatrix}},\quad \frac{\begin{bmatrix}\eta_2\ \eta_3\\ h_1\ h_4\end{bmatrix}\times\begin{bmatrix}\eta_1\ \eta_4\\ h_2\ h_3\ h_5\ h_6\ h_7\end{bmatrix}}{\begin{bmatrix}h_2\ h_4\ h_5\ h_6\ h_7\\ h_2\ h_3\end{bmatrix}\times\begin{bmatrix}\eta_2\ \eta_3\\ \eta_1\ \eta_4\end{bmatrix}}.$$

Any other term will necessarily contain in the numerator a factor, whose symbolical representation will contain one of the quantities $\eta_1\ \eta_2\ \eta_3\ \eta_4$ in the upper line, and one of the quantities $h_1\ h_2\ h_3\ h_4$, having the same subscript index in the lower line, and which will therefore vanish; the number of effective terms being evidently the number of ways in which four things can be combined 2 and 2 together, and the value of each term is evidently $(-)^{2.3}.(-1)^{2.5}.1$, so that the entire value of the coefficient of $\eta_5^2.\eta_6^2$ in $\Theta_{2,2}$ is $+6$.

Precisely in the same manner, we shall find that the leading coefficient in $\Theta_{3,1}$ will contain the term $-4\eta_5^3.\eta_6^3$, the (-1) resulting from the operation $(-1)^{1.3}.(-1)^{3.4}$, and in $\Theta_{4,0}$ the term $+\eta_5^4.\eta_6^4$, the $+1$ resulting from the operation $(-1)^{4.3}$. Hence it appears that $\Theta_{0,4}$; $\Theta_{1,3}$; $\Theta_{2,2}$; $\Theta_{3,1}$; $\Theta_{4,0}$ are to one another in the ratios of 1; -4; 6; -4; 1; and so in general for any values of m, n, i (i being less than m and less than n) it will be found that

$$\Theta_{0,i},\quad \Theta_{1,i-1},\quad \Theta_{2,i-2},\ \ldots\Theta_{i,0}$$

will be in the ratios of the numbers

$$1;\quad (-1)^{m-1}.i;\quad (-1)^{2(m-2)}.i.\frac{i-1}{2};\quad (-1)^{3(m-3)}.i.\frac{i-1}{2}.\frac{i-2}{3},\ \ldots;\quad (-1)^{i(m-i)}.$$

Art. (27.). The method employed in the preceding investigation will enable us to affix the proper sign and numerical factor to $\Theta_{0,i}$ or $\Theta_{i,0}$, or in general to $\Theta_{\theta,i-\theta}$, in order that it may represent the Bezoutian secondary of the degree i in x. [This latter has been already identified with the simplified residue obtained by expanding $\frac{\phi x}{fx}$ under the form of an improper continued fraction.] For this purpose, it will be sufficient to compare a single term of any such Θ with the corresponding one in the Symmorphic Bezoutian secondary. Let us first suppose that $m=n$, f and ϕ being of

the same degree. A glance at the form of the Bezoutian square will show that if we form the Bezoutian secondary of the degree $(n-i)$ in x, the coefficient of its leading term will contain the term $(-)^{(i-1)\frac{i}{2}}(0,\ i)^i$; $(0,\ i)$ as usual denoting the product of the coefficient of x^n in f by the coefficient of x^{n-i} in φ, less the product of the coefficient of x^n in φ by that of x^{n-i} in f; and as we suppose the first coefficients in f and φ to be each 1, if we term the other coefficients last spoken of a_i and α_i respectively, this said coefficient of the leading term of the ith Bezoutian secondary will contain the term $(-)^{(i-1)\frac{i}{2}}(a_i-\alpha_i)^i$, and consequently $(-1)^{(i-1)\frac{i}{2}}.\alpha_i^i$ and $(-)^{i\frac{i+1}{2}}a_i^i$.

Now by the like reasoning as that employed in the preceding article, the coefficient of the leading term in $\mathfrak{B}_{m-i,\,0}$ *i. e.*

$$\Sigma(x-h_{q_{i+1}})(x-h_{q_{i+2}})\ldots(x-h_{q_m})\frac{\begin{bmatrix} h_{q_1}\ h_{q_2}\ldots h_{q_i} \\ \eta_1\ \ \eta_2\ \ldots\eta_m \end{bmatrix}}{\begin{bmatrix} h_{q_1}\ \ h_{q_2}\ \ \ldots h_{q_i} \\ h_{q_{i+1}}\ h_{q_{i+2}}\ldots h_{q_m} \end{bmatrix}}$$

will contain the quantity $\Sigma(h_1.h_2.h_3\ldots h_i)^i$, and therefore will contain a term $\left(\Sigma(h_1.h_2.h_3\ldots h_i)\right)^i$, *i. e.* $(-)^{ii}a_i^i$, which is equal to $(-)^i a_i^i$, since $(i-1)i$ is always even. Hence $\mathfrak{B}_{m-i,\,0}=(-)^{i\cdot\frac{i-1}{2}}\times$ the corresponding Bezoutian secondary.

Art. (28.). The above applies to the case where we have supposed $m=n$. When this equality does not exist we may proceed as follows. Prefix to $\varphi(x)$, the first coefficient of which is still supposed to be 1, a term $\varepsilon-x^m$, where ε is positive and indefinitely small, and let φx so augmented be called $\Phi(x)$. Then if $k_1k_2\ldots k_n$ are the roots of φx, $k_1k_2\ldots k_n$, together with the $(m-n)$ values of $\left(\frac{1}{\varepsilon}\right)^{\frac{1}{m-n}}$, will be the roots of $\Phi(x)$.

But it has already been proved that when (as here supposed) the first coefficient of fx is 1, the Bezoutian secondaries to f and φ will be identical with those to f and Φ respectively; at least it has been proved that these latter, when $\varepsilon=0$, but the form of Φ is preserved, become identical with the former, and consequently the same is true when ε is taken indefinitely small. Now if we call the $(m-n)$ roots of Φ which do not belong to φ, $\eta_{n+1},\ \eta_{n+2}\ldots\eta_m$, and make

$$\Psi_{m-i,\,0}=\Sigma(x-h_{q_{i+1}})(x-h_{q_{i+2}})\ldots(x-h_{q_m})\cdot\frac{\begin{bmatrix} h_{q_1}\ h_{q_2}\ldots h_{q_i} \\ \eta_1\ \ \eta_2\ \ldots\eta_m \end{bmatrix}}{\begin{bmatrix} h_{q_1}\ \ h_{q_2}\ \ \ldots h_{q_i} \\ h_{q_{i+1}}\ h_{q_{i+2}}\ldots h_{q_m} \end{bmatrix}},$$

we have $\Psi_{m-i,\,0}=\Sigma\mathrm{P}(h_{q_1}h_{q_2}\ldots h_{q_i})\begin{bmatrix} h_{q_1}\ \ h_{q_2}\ \ \ldots h_{q_i} \\ \eta_{n+1}\ \eta_{n+2}\ldots\eta_m \end{bmatrix}$,

where $\mathrm{P}(h_{q_1}h_{q_2}\ldots h_{q_i})=(x-h_{q_{i+1}})(x-h_{q_{i+2}})\ldots(x-h_{q_m})\dfrac{\begin{bmatrix} h_{q_1}\ h_{q_2}\ldots h_{q_i} \\ \eta_1\ \ \eta_2\ \ldots\eta_n \end{bmatrix}}{\begin{bmatrix} h_{q_1}\ \ h_{q_2}\ \ \ldots h_{q_i} \\ h_{q_{i+1}}\ h_{q_{i+2}}\ldots h_{q_m} \end{bmatrix}}$.

But since $k_{n+1}, k_{n+2}\ldots k_m$ are infinite in value,

$$\begin{bmatrix} h_{q_1} & h_{q_2} & \ldots h_{q_i} \\ k_{n+1} & k_{n+2} & \ldots k_m \end{bmatrix} = \left((-k_{n+1}).(-k_{n+2})\ldots(-k_m)\right)^i \left(\frac{1}{s}\right)^i.$$

Hence

$$\Psi_{m-i,\,0} = \left(\frac{1}{s}\right)^i \Sigma P(h_{q_1} h_{q_2} \ldots h_{q_i})$$

$$= \left(\frac{1}{s}\right)^i \mathfrak{S}_{m-i,\,0}$$

and

$$\mathfrak{S}_{m-i,\,0} = s^i \Psi_{m-i,\,0}.$$

But by what has been shown antecedently [taking account of the fact of the leading coefficient of Φ being s in place of 1, which introduces the factor s^i], we have

$$s^i \Psi_{m-i,\,0} = (-)^{(i-1)\frac{i}{2}} B'_i,$$

where B'_i is the Bezoutian secondary of the $(m-i-1)$th degree in x to f and φ; but B'_i it has been proved $= B_i$, the Bezoutian secondary of the same degree to f and φ; hence $\mathfrak{S}_{m-i,\,0} = (-)^{i \cdot \frac{i-1}{2}} . B_i$.

Art. (29.). If now we return to the syzygetic equation, $\tau f - t\varphi + \mathfrak{S} = 0$, $\mathfrak{S}$ may be treated as known, having in fact been completely determined as a function of the roots, as well in its most general form, as also so as to represent the simplified residues to f and φ in the preceding articles; it remains to determine the values of τ and t as functions of the roots corresponding to any allowable form of $\mathfrak{S}$, but I shall confine the investigation to the case where $\mathfrak{S}$ is the lowest-weighted conjunctive, or which is the same thing, a simplified residue to t and φ of any given degree in x; each value of $\frac{\tau}{t}$ will then represent one of the convergents to $\frac{\varphi}{f}$ when expanded under the form of a continued fraction. If $\mathfrak{S}$ be of the ith degree in x, τ is of the degree $(n-i-1)$ and t of the degree $(m-i-1)$. This being supposed, and calling $n-i-1=\nu$, $m-i-1=\mu$, I say that t will be represented by G and τ by Γ, where

$$G = (-)^i \Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_\mu}) \frac{\begin{bmatrix} h_{q_1} & h_{q_2} & \ldots h_{q_m} \\ \eta_1 & \eta_2 & \ldots \eta_n \end{bmatrix}}{\begin{bmatrix} h_{q_1} & h_{q_2} & \ldots h_{q_\mu} \\ h_{q_{\mu+1}} & h_{q_{\mu+2}} & \ldots h_{q_m} \end{bmatrix}},$$

and τ is an analogous form Γ; $h_1, h_2 \ldots h_m$, as heretofore, being the roots of f, and $\eta_1, \eta_2 \ldots \eta_n$ of φ. To fix the ideas and make the demonstration more immediately seizable, give m and n specific values; thus let $m=5$, $n=4$, $i=2$, so that $\mu = 5-2-1=2$. Put $\mathfrak{S}$ under the form $\mathfrak{S}_{i,\,0}$, so that $\mathfrak{S}$ in the case before us

$$= \Sigma(x-h_{q_1})(x-h_{q_2}) \frac{\begin{bmatrix} h_{q_2} & h_{q_4} & h_{q_5} \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{bmatrix}}{\begin{bmatrix} h_{q_2} & h_{q_3} & h_{q_5} \\ h_{q_1} & h_{q_3} \end{bmatrix}}.$$

Now make $x=h$, then $f=0$, and $\mathfrak{S}$ becomes

$$\Sigma(h_1-h_{q_1})(h_1-h_{q_2})\frac{\begin{bmatrix} h_{q_3} h_{q_4} h_{q_5} \\ \eta_1\ \eta_2\ \eta_3\ \eta_4 \end{bmatrix}}{\begin{bmatrix} h_{q_2} h_{q_4} h_{q_5} \\ h_{q_1} h_{q_3} \end{bmatrix}},$$

$$i.\,e.\ \Sigma\frac{\begin{bmatrix} h_1 \\ h_4 h_5 \end{bmatrix}\begin{bmatrix} h_1 h_2 h_3 \\ \eta_1\ \eta_2\ \eta_3\ \eta_4 \end{bmatrix}}{\begin{bmatrix} h_1 h_2 h_3 \\ h_4 h_5 \end{bmatrix}},$$

h_1 being kept constant in the above sum, but h_2, h_3, h_4, h_5 being partitionable in all the six possible ways into two groups, as into h_4, h_5, h_2, h_3 in the term above expressed. This sum is evidently identical with

$$\Sigma\frac{\begin{bmatrix} h_1 h_2 h_3 \\ \eta_1\ \eta_2\ \eta_3\ \eta_4 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 \\ h_4 h_5 \end{bmatrix}},\ i.\,e.\ \begin{bmatrix} h_1 \\ \eta_1\ \eta_2\ \eta_3\ \eta_4 \end{bmatrix}\times\Sigma\frac{\begin{bmatrix} h_2 h_3 \\ \eta_1\ \eta_2\ \eta_3\ \eta_4 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 \\ h_4 h_5 \end{bmatrix}}.$$

Again, φ becomes

$$\begin{bmatrix} h_1 \\ \eta_1\ \eta_2\ \eta_3\ \eta_4 \end{bmatrix}.$$

Hence $t=\frac{\mathfrak{S}}{\varphi}$ becomes

$$\Sigma\frac{\begin{bmatrix} h_2 h_3 \\ \eta_1\ \eta_2\ \eta_3\ \eta_4 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 \\ h_4 h_5 \end{bmatrix}}.$$

But when $x=h_1$, $\frac{G}{(-)^i}$ becomes

$$\begin{bmatrix} h_1 \\ h_2 h_3 \end{bmatrix}\frac{\begin{bmatrix} h_2 h_3 \\ \eta_1\ \eta_2\ \eta_3\ \eta_4 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 \\ h_1 h_4 h_5 \end{bmatrix}},$$

$$i.\,e.\ =\frac{\begin{bmatrix} h_1 \\ h_2 h_3 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 \\ h_1 \end{bmatrix}}\cdot\frac{\begin{bmatrix} h_2 h_3 \\ \eta_1\ \eta_2\ \eta_3\ \eta_4 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 \\ h_4 h_5 \end{bmatrix}},$$

$$=(-1)^i.t.$$

Thus when $x=h_1$, $t=G$. In like manner, when $x=h_2$, or h_3, or h_4, or h_5, t always $=G$; but t and G are both functions of x of the same degree, and of only two dimensions in x. Hence t is identical with G. So in general it may be proved, that whenever $x=h_1$, or h_2 or h_3,... or h_n, t and G, which are each of only $(n-1-i)$ dimensions in x,

are equal. Hence universally $t=\mathrm{G}$, as was to be shown. To find τ we must avail ourselves of the symmorphic, or as we may better say (it being at the opposite extremity of the scale of forms, the antimorphic), value of $\mathfrak{S}$ represented by $\mathfrak{S}_{0,b}$ taking care to preserve $\mathfrak{S}$ strictly identical under both forms of representation, in point of sign as well as quantity. That is to say, we must make

$$\mathfrak{S}_{0,i}=(-)^{i(m-i)}\Sigma(x-\eta_{q_1})(x-\eta_{q_2})\ldots(x-\eta_{q_i})\frac{\begin{bmatrix} h_1 & h_2 & \ldots h_m \\ \eta_{q_{i+1}} & \eta_{q_{i+2}} & \ldots h_{q_n} \end{bmatrix}}{\begin{bmatrix} \eta_{q_{i+1}} & \eta_{q_{i+2}} & \ldots \eta_{q_n} \\ \eta_{q_1} & \eta_{q_2} & \ldots \eta_{q_i} \end{bmatrix}}$$

$$=(-)^{\omega}\Sigma(x-\eta_{q_1})(x-\eta_{q_2})\ldots(x-\eta_{q_i})\frac{\begin{bmatrix} \eta_{q_{i+1}} & \eta_{q_{i+2}} & \ldots \eta_{q_n} \\ h_1 & h_2 & \ldots h_m \end{bmatrix}}{\begin{bmatrix} \eta_{q_{i+1}} & \eta_{q_{i+2}} & \ldots \eta_{q_n} \\ \eta_{q_1} & \eta_{q_2} & \ldots \eta_{q_i} \end{bmatrix}},$$

where $$\omega=i(m-i)+m(n-i),$$

so that $$(-)^{\omega}=(-)^{mi-i+mn-mi}=(-)^{mn-i};$$

and consequently the same reasoning as was applied to t to prove $t=\mathrm{G}$, will serve to show that $-\tau=\Gamma$, where

$$\Gamma=(-)^{mn}.\Sigma(x-\eta_{\xi_1})(x-\eta_{\xi_2})\ldots(x-\eta_{\xi_\nu})\frac{\begin{bmatrix} \eta_{\xi_1} & \eta_{\xi_2} & \ldots \eta_{\xi_\nu} \\ h_1 & h_2 & \ldots h_m \end{bmatrix}}{\begin{bmatrix} \eta_{\xi_1} & \eta_{\xi_2} & \ldots \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}} & \eta_{\xi_{\nu+2}} & \ldots \eta_{\xi_n} \end{bmatrix}},$$

or

$$\tau=(-)^{\omega}\Sigma(x-\eta_{\xi_1})(x-\eta_{\xi_2})\ldots(x-\eta_{\xi_\nu})\frac{\begin{bmatrix} h_1 & h_2 & \ldots h_m \\ \eta_{\xi_1} & \eta_{\xi_2} & \ldots \eta_{\xi_\nu} \end{bmatrix}}{\begin{bmatrix} \eta_{\xi_1} & \eta_{\xi_2} & \ldots \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}} & \eta_{\xi_{\nu+2}} & \ldots \eta_{\xi_n} \end{bmatrix}},$$

where

$$\omega=mn-1-m\nu=mn-1-m(n-i-1)$$
$$=mi-m-1.$$

Art. (30.). I have not succeeded in throwing t and τ under any other than the single forms for each above given, and it is remarkable that whilst apparently t and τ admit only of this single representation, $\mathfrak{S}$ admits of the variety of forms included under the general symbol $\mathfrak{S}_{n,i-\nu}$ for a given value of i; and it ought to be remarked that these forms (although the most perfectly symmetrical and exactly balanced representations) [and for that reason possibly the most commodious for the ascertainment of the allotrious factor belonging to them respectively] by no means exhaust the almost infinite variety of modes by which the simplified residues, *i. e.* the hekistobarytic, or if we like so to call them, the prime conjunctives, admit of being represented

as functions of the roots of the given functions; but if in art. (16.), instead of writing

$$R=\frac{\begin{bmatrix} h_{q_1} h_{q_2}\dots h_{q_\nu} \\ \eta_{\xi_1} \eta_{\xi_2}\dots\eta_{\xi_\nu}\end{bmatrix}}{\begin{bmatrix} h_{q_1} & h_{q_2} & \dots h_{q_\nu} \\ h_{q_{\nu+1}} & h_{q_{\nu+2}}\dots h_{q_m}\end{bmatrix}\times\begin{bmatrix} \eta_{\xi_1} & \eta_{\xi_2} & \dots\eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}} & \eta_{\xi_{\nu+2}}\dots\eta_{\xi_n}\end{bmatrix}},$$

we had made

$$R=\frac{P(h_{q_1} h_{q_2}\dots h_{q_\nu};\ \eta_{\xi_1}\eta_{\xi_2}\dots\eta_{\xi_\nu})}{\begin{bmatrix} h_{q_1} & h_{q_2} & \dots h_{q_\nu} \\ h_{q_{\nu+1}} & h_{q_{\nu+2}}\dots h_{q_m}\end{bmatrix}\times\begin{bmatrix} \eta_{\xi_1} & \eta_{\xi_2} & \dots\eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}} & \eta_{\xi_{\nu+2}}\dots\eta_{\xi_m}\end{bmatrix}},$$

where P represents any function symmetrical in respect of $h_{q_1}, h_{q_2}\dots h_{q_\nu}$, and also in respect of $\eta_{\xi_1}, \eta_{\xi_2}\dots\eta_{\xi_\nu}$ (the interchanges, that is to say, between one h and another h, or between one η and another η, leaving P unaltered), it might be shown that the value of $\vartheta_{v,\nu}$ resulting from the introduction of this more general value of R would (as for the particular value assumed) always be expressible as an integral function of the roots, and consequently, if P be taken of the same dimensions in the roots as the numerator of R previously assumed, *i. e.* $v\nu$, $\vartheta_{v,\nu}$ would continue to be (unless indeed it vanish) identical (to some numerical factor *près*) with the corresponding simplified residue. If, on the other hand, P be taken of less than $v\nu$ dimensions in h and k, we know *à priori* that $\vartheta_{v,\nu}$ must vanish, as otherwise we should have a conjunctive of a weight less than the minimum weight. When P is of the proper *amount* of weight $v\nu$, it is I think probable that another condition as to the *distribution* of the weight will be found to be necessary in order that $\vartheta_{v,\nu}$ may not vanish, viz. that the highest power of any single (h) in P shall not exceed v, nor the highest power of any single η exceed ν. But as I have not had leisure to enter upon the inquiry, the verification or disproval of this supposed law, and more generally the evolution of the allotrious numerical factor introduced into $\vartheta_{v,\nu}$ by assigning any particular form to (P) satisfying the necessary conditions of amount and distribution of weight, must be reserved, amongst other points connected with the theory of the remarkable forms (19.) art. (15.), as a subject for future investigation.

Art. (31.). A property of continued fractions, which, if known, I have not met with in any treatise on the subject (but which has been already cursorily alluded to in these pages), gives rise to a remarkable property of reciprocity connecting τ and t severally with ϑ in the syzygetic equation $\tau f - t\phi + \vartheta = 0$.

Let the successive convergents to the ordinary continued fraction

$$\frac{1}{q_1+}\ \frac{1}{q_2+}\ \frac{1}{q_3+}\cdots\frac{1}{q_{i-1}+}\ \frac{1}{q_i}$$

be called

$$\frac{l_1}{m_1},\ \frac{l_2}{m_2}\cdots\frac{l_{i-1}}{m_{i-1}}\ \frac{l_i}{m_i}$$

respectively, it is well known that

$$m_{i-1}.l_i - m_i\, l_{i-1} = (-)^{i-1}.1\,;$$

but I believe that it has not been observed that this is only the extreme cases of a much more general equation, viz.

$$m_{i-\rho}\, l_i - m_i\, l_{i-\rho} = (-)^{i-\rho}.\mu_{\rho-1}\text{*},$$

where $\mu_1, \mu_2, \ldots \mu_i$ denote respectively the denominators to the convergents to the continued fractions formed with the quotients taken in a reverse order, *i. e.* the continued fraction

$$\frac{1}{q_i+}\ \frac{1}{q_{i-1}+}\ \frac{1}{q_{i-2}+}\cdots+\frac{1}{q_2+}\ \frac{1}{q_1}.$$

This is easily proved when $\rho=1$; μ_0 is of course (as usual) to be considered 1. So more simply for the improper continued fraction,

$$\frac{l_i}{m_i}=\frac{1}{q_1-}\ \frac{1}{q_2-}\cdots\frac{1}{q_{i-1}-}\ \frac{1}{q_i},$$

of which the convergents are supposed to be

$$\frac{l_1}{m_1}\ \frac{l_2}{m_2}\cdots\frac{l_{i-1}}{m_{i-1}}\ \frac{l_i}{m_i},$$

and the reverse fraction

$$\frac{1}{q_i-}\ \frac{1}{q_{i-1}-}\cdots\frac{1}{q_2-}\ \frac{1}{q_1},$$

of which the convergents are supposed to be

$$\frac{\lambda_1}{\lambda_2},\ \frac{\lambda_3}{\mu_2},\cdots\frac{\lambda_i}{\mu_i},$$

we have the more simple equation

$$l_i.m_{i-\rho}-l_{i-\rho}.m_i+\mu_{\rho-1}=0.$$

And it is well known, or at all events easily demonstrable, that

$$\frac{l_{i-1}}{l_i}=\frac{1}{q_i-}\ \frac{1}{q_{i-1}-}\ \frac{1}{q_{i-2}-}\cdots\frac{1}{q_2}$$

$$\frac{m_{i-1}}{m_i}=\frac{1}{q_i-}\ \frac{1}{q_{i-1}-}\ \frac{1}{q_{i-2}-}\cdots\frac{1}{q_2}\ \frac{1}{q_1}.$$

Art. (32.). If now we use subscript indices to denote the degree in x of the quantities to which they are affixed, we have the general syzygetic equation

$$\mathrm{K}\tau_{m-i-1}f_m-\mathrm{K}t_{m-i-1}.\varphi_n+\mathrm{K}\mathfrak{Y}_i=0,$$

where K, a constant (which I have given the means of determining in the first section), being rightly assumed $\mathrm{K}.\tau_{n-i-1}$, $\mathrm{K}\tau_{m-i-1}$, become the numerator and denominator respectively of one of the convergents to $\frac{\varphi}{f}$, expressed as an improper continued fraction, and $\mathrm{K}\mathfrak{Y}_i$ becomes the denominator to one of the convergents to $\frac{t^{m-1}}{f}$, or,

* See London and Edinburgh Philosophical Magazine, "On a Fundamental Theorem in the Theory of Continued Fractions," October, 1853.

which is the same thing, to $\frac{\tau^{n-1}}{\varphi}$*. Conversely, it is obvious that if we adopt as our primitive functions $cf(m)$ and t_{m-1}, (c) being the value of K when $i=0$, we shall obtain as the general form of our syzygetic equation, bearing in mind that $(m-1)$ now replaces (n),

$$c.\mathrm{K}'(\tau)_{m-i}fm-\mathrm{K}'\mathfrak{S}_{m-i-1}t_{m-1}+\mathrm{K}'\tau_i=0;$$

and similarly, if we adopt as our primitive functions τ_{n-1} and $c\varphi_n$, we obtain for our general syzygetic equation, observing that $(n-1)$ now replaces (m),

$$\mathrm{K}'\mathfrak{S}_{n-i-1}.\tau_{n-1}-c\mathrm{K}'.\mathfrak{S}_{n-i-1}.\varphi_n+\mathrm{K}'\tau_i=0;$$

so that (making abstraction of the constant factors and looking merely to the forms of the several functions which enter into the equations) we see that on the first hypothesis, viz. of t_{m-1} being substituted for φ_n, the conjunctives of each degree in x change places with the second conjunctive factors, *i. e.* the original multipliers of φ of the same degree in x, and *vice versâ*; and in the second hypothesis, where τ_{n-1} takes the place of fm, the conjunctives of each degree in x change places with the first conjunctive factors, *i. e.* the original multipliers of f of the same degree in x, and *vice versâ*; t_{m-1} and τ_{n-1} being respectively multipliers of φ and f, such that the difference of the respective products is independent of x. These results ought to be capable of being verified by aid of our general formulæ for t, τ, $\mathfrak{S}$, and as this verification will serve to exhibit in a clearer light the nature of the reciprocity between the conjunctives and the conjunctive factors, it may be not uninteresting to set it out.

Art. (33.). As usual, let $h_1 h_2 \ldots h_m$ be the roots of $f(x)$, and $\eta_1 \eta_2 \ldots \eta_{m-1}$ the roots of $\varphi(x)$, the last conjunctive factor to ϕ, which is of the degree $(m-1)$ in x, will be represented, neglecting powers of $(-)$, by t_{m-1}, where

$$t_{m-1}=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_{m-1}})\frac{\begin{bmatrix} h_{q_1} h_{q_2}\ldots h_{q_{m-1}} \\ \eta_1\ \eta_2\ \ldots\eta_{q_n} \end{bmatrix}}{\begin{bmatrix} h_{q_m} \\ h_{q_1} h_{q_2}\ldots h_{q_{m-1}} \end{bmatrix}}.$$

If now we for greater simplicity make $t_{n-1}=t(x)$, and call the roots of t, $\eta'_1 \eta'_2 \ldots \eta'_{n-1}$ any such quantity as

$$\begin{bmatrix} h_{q_m} \\ \eta'_1\ \eta'_2\ldots\eta'_{m-1} \end{bmatrix}=t(h_{q_m})=(h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})\ldots(h_{q_m}-h_{q_{m-1}})\times\frac{\varphi(h_{q_1}).\varphi(h_{q_2})\ldots\varphi(h_{q_{m-1}})}{(h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})\ldots(h_{q_m}-h_{q_{m-1}})}$$

$$=\varphi(h_{q_1})\varphi(h_{q_2})\ldots\varphi(h_{q_{m-1}})$$

$$=\mathrm{R}\frac{1}{\varphi(h_{q_m})},$$

* Since i is always supposed less than n (n being the degree of the lower degreed of the two functions f and φ), the fact of the last quotient to $\frac{t_{m-1}}{f}$ being wanting to $\frac{\tau_{n-1}}{\varphi}$ will not affect the accuracy of the statement in the text above, since this latter will contain as many quotients as can in any case be required for expressing $\mathfrak{S}_i$.

R denoting a constant independent of the root h_{q_m} selected (and which constant is in fact the resultant of the two functions $f(x)$ and $\phi(x)$), that is to say,

$$\phi(h_1)\phi(h_2)\phi(h_3)\ldots\phi(h_m).$$

But by our general formulæ (8.) the simplified residue to $f(x)$ and $t(x)$ of the ith degree in x will be represented by

$$\mathfrak{S}'_{i,0}=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i})\left\{\frac{\begin{bmatrix} h_{q_{i+1}} h_{q_{i+2}}\ldots h_{q_m} \\ \eta'_1 \quad \eta'_2 \quad \ldots\eta'_{m-1} \end{bmatrix}}{\begin{bmatrix} h_{q_1} \quad h_{q_2} \quad \ldots h_{q_i} \\ h_{q_{i+1}} h_{q_{i+2}}\ldots h_{q_m} \end{bmatrix}}\right\};$$

$$\therefore\ \mathfrak{S}'_{i,0}=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i})\times\left\{\frac{\mathrm{R}^{m-i}.(\phi h_{q_{i+1}})^{-1}.\phi(h_{q_{i+2}})^{-1}\ldots\phi(h_{q_m})^{-1}}{\begin{bmatrix} h_{q_1} \quad h_{q_2} \quad \ldots h_{q_i} \\ h_{q_{i+1}} h_{q_{i+2}}\ldots h_{q_m} \end{bmatrix}}\right\}$$

$$=\mathrm{R}^{m-i-1}\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i})\frac{\phi(h_{q_1})\phi(h_{q_2})\ldots\phi(h_{q_i})}{\begin{bmatrix} h_{q_1} \quad h_{q_2} \quad \ldots h_{q_i} \\ h_{q_{i+1}} h_{q_{i+2}}\ldots h_{q_m} \end{bmatrix}},$$

or $\mathfrak{S}'_i=\mathrm{R}^{m-i-1}t_i$,

the relation which was to be obtained. So conversely, in precisely the same manner, calling t'_i the conjunctive factor of the degree i in x to $t(x)$ in the syzygetic equation, which connects $f(x)$ and $t(x)$ with a corresponding simplified residue, we have

$$t'_i=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i})\frac{\begin{bmatrix} h_{q_1} h_{q_2}\ldots h_{q_i} \\ \eta'_1 \quad \eta'_2 \quad \ldots\eta'_{m-1} \end{bmatrix}}{\begin{bmatrix} h_{q_1} \quad h_{q_2} \quad \ldots h_{q_i} \\ h_{q_{i+1}} h_{q_{i+2}}\ldots h_{q_m} \end{bmatrix}}$$

$$=\mathrm{R}^{i-1}\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i})\frac{\phi h_{q_{i+1}}\phi h_{q_{i+2}}\ldots\phi h_{q_m}}{\begin{bmatrix} h_{q_1} \quad h_{q_2} \quad \ldots h_{q_i} \\ h_{q_{i+1}} h_{q_{i+2}}\ldots h_{q_m} \end{bmatrix}}$$

$$=\mathrm{R}^{i-1}.\mathfrak{S}_i,$$

the conjugate equation to the one previously obtained*.

And evidently the same reasoning serves to establish the reciprocity, or rather reciprocal convertibility, between the $\mathfrak{S}$ series and the τ series, when in lieu of the original primitives $f(x)$ and $\phi(x)$ we take as our primitives $\tau(x)$ and $\phi(x)$, $\tau(x)$ being the function which satisfies the equation

$$\tau(x)fx-t(x)\phi x+\mathrm{R}=0.$$

Art. (34.). It may be remarked that if $n=m-1$ (the last syzygetic equation being

* M. Hermite, by a peculiar method, first discovered one of these two conjugate relations of reciprocity, applicable to the case of Sturm's theorem, where $\phi x=f'x$, and I am indebted to him for bringing the subject under my notice.

thus $t_{m-1}.\varphi_{m-1}-\tau_{m-2}.f_m+S_0=0$), when t_{m-1} and f_m are taken as the primitives, the corresponding equation will be of the form

$$t'_{m-1}.t_{m-1}-\tau'_{m-2}f_m+S'_0=0;$$

these two equations must therefore be identical, and consequently $t'_{m-1}=\varphi_{m-1}$ (to a numerical factor *près*), so that t_{m-1} and φ_{m-1} are reciprocal forms; this is also obvious from the consideration that t'_{m-1} must, by the general law of reciprocity (established above), be a residue to (f_m, φ_{m-1}), which the latter function itself may be considered to be. Or the same thing is obvious directly, by writing

$$t_{m-1}=t(x)=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_{m-1}}).\frac{\varphi(h_{q_1})\varphi(h_{q_2})\ldots\varphi(h_{q_{m-1}})}{(h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})\ldots(h_{q_m}-h_{q_{m-1}})},$$

and then making

$$t'_{m-1}=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_{m-1}}).\frac{t(h_{q_1})t(h_{q_2})\ldots t(h_{q_{m-1}})}{(h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})\ldots(h_{q_m}-h_{q_{m-1}})}$$

$$=\Sigma(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_{m-1}}).\frac{(\varphi(h_{q_1}).\varphi(h_{q_2})\ldots\varphi h_{q_{m-1}}.h_{q_m})}{\Delta},$$

where

$$\Delta=(-)^{m.\frac{m-1}{2}}(h_1-h_2)^2.(h_1-h_3)^2\ldots(h_1-h_m)^2$$
$$\times(h_2-h_3)^2\ldots(h_2-h_m)^2$$
$$\ldots\ldots\ldots$$
$$\times(h_{m-1}-h_m)^2$$

$$=(-1)^{m.\frac{m-1}{2}}\mathrm{D}$$ (D being the Discriminant, more commonly called the Determinant to f); or finally,

$$t'_{m-1}=\frac{\mathrm{R}^{m-1}}{\mathrm{D}}\varphi,$$ as was to be shown.

Section III.

On the application of the Theorems in the preceding Section to the expression in terms of the roots of any primitive function of Sturm's *auxiliary functions, and the other functions which connect these with the primitive function and its first differential derivative.*

Art. (35.). The formulæ in the preceding Section had reference to the case of two absolutely independent functions and their respective systems of roots: when the functions become so related that the roots of the one system become explicitly or implicitly functions of the roots of the other system, the formulæ will become expressible in terms of these latter alone, and in some cases the terms (of which the sum is always essentially integral) will become separately and individually representable under an integral form. Such, as I shall proceed to show, is the case for two functions, of which one is the differential derivative of the other. When f and φ are

thus related, so that $\varphi=\frac{df}{dx}$, calling as before $h_1\ h_2 \ldots h_m$ the roots of f, and $\eta_1\ \eta_2 \ldots \eta_{m-1}$ the roots of φ, we shall have in general

$$\begin{bmatrix} h_{q_{i+1}} \\ \eta_1\ \eta_2 \ldots \eta_{m-1} \end{bmatrix} = (h_{q_{i+1}}-\eta_1)(h_{q_i}-\eta_2)\ldots(h_{q_{i+1}}-\eta_{m-1})$$

$$= f' h_{q_{i+1}} = \begin{bmatrix} h_{q_{i+1}} \\ h_{q_1} h_{q_2} \ldots h_{q_i} h_{q_{i+2}} \ldots h_{q_m} \end{bmatrix} = \begin{bmatrix} h_{q_{i+1}} \\ h_{q_1} h_{q_2} \ldots h_{q_i} \end{bmatrix} \times \begin{bmatrix} h_{q_{i+1}} \\ h_{q_{i+2}}\ h_{q_{i+3}} \ldots h_{q_m} \end{bmatrix}.$$

Consequently

$$\begin{bmatrix} h_{q_{i+1}}\ h_{q_{i+2}} \ldots h_{q_m} \\ \eta_1 \quad \eta_2 \quad \ldots \eta_{m-1} \end{bmatrix} = \begin{bmatrix} h_{q_{i+1}} \\ \eta_1 \eta_2 \ldots \eta_{m-1} \end{bmatrix} \times \begin{bmatrix} h_{q_{i+2}} \\ \eta_1 \quad \eta_2 \ldots \eta_{m-1} \end{bmatrix} \times \&c. \times \begin{bmatrix} h_{q_m} \\ \eta_1 \quad \eta_2 \ldots \eta_{m-1} \end{bmatrix}$$

$$= \begin{bmatrix} h_{q_{i+1}} \\ h_{q_1} h_{q_2} \ldots h_{q_i} \end{bmatrix} \times \begin{bmatrix} h_{q_{i+1}} \\ h_{q_{i+2}}\ h_{q_{i+3}} \ldots h_{q_{m-1}} \end{bmatrix}$$

$$\times \begin{bmatrix} h_{q_{i+2}} \\ h_{q_1} h_{q_2} \ldots h_{q_i} \end{bmatrix} \times \begin{bmatrix} h_{q_{i+2}} \\ h_{q_{i+1}}\ h_{q_{i+3}} \ldots h_{q_{m-1}} \end{bmatrix}$$

$$\times \&c. \quad . \quad . \quad . \quad .$$

$$\times \begin{bmatrix} h_{q_m} \\ h_{q_1} h_{q_2} \ldots h_{q_i} \end{bmatrix} \times \begin{bmatrix} h_{q_m} \\ h_{q_{i+1}}\ h_{q_{i+2}} \ldots h_{q_{m-1}} \end{bmatrix}.$$

Hence

$$\frac{\begin{bmatrix} h_{q_{i+1}}\ h_{q_{i+2}} \ldots h_{q_m} \\ \eta_1 \eta_2 \quad \ldots \eta_{m-1} \end{bmatrix}}{\begin{bmatrix} h_{q_{i+1}}\ h_{q_{i+2}} \ldots h_{q_m} \\ h_{q_1} \quad h_{q_2} \quad \ldots h_{q_i} \end{bmatrix}} =$$

$$\begin{bmatrix} h_{q_{i+1}} \\ h_{q_{i+2}}\ h_{q_{i+3}} \ldots h_{q_m} \end{bmatrix} \times \begin{bmatrix} h_{q_{i+2}} \\ h_{q_i} \quad h_{q_{i+1}} \ldots h_{q_m} \end{bmatrix} \times \ldots\ldots\ldots \times \begin{bmatrix} h_{q_m} \\ h_{q_1}\ h_{q_2} \ldots h_{q_{m-1}} \end{bmatrix}$$

$$= (-)^{\frac{1}{2}(m-i)(m-i-1)} . \zeta(h_{q_{i+1}}\ h_{q_{i+2}} \ldots h_{q_m}),$$

the ζ denoting the operation of taking the product of the squares of the differences of the quantities which this symbol governs. Hence the Bezoutian secondary to f and f' of the $(m-i-1)$th degree in x, viz.—

$$(-)^{i \cdot \frac{i-1}{2}} \Sigma(x-h_{q_{i+1}})(x-h_{q_{i+2}}) \ldots (x-h_{q_m}) \frac{\begin{bmatrix} h_{q_1} h_{q_2} \ldots h_{q_i} \\ \eta_1 \quad \eta_2 \ldots \eta_{m-1} \end{bmatrix}}{\begin{bmatrix} h_{q_1} \quad h_{q_2} \quad \ldots h_{q_i} \\ h_{q_{i+1}} h_{q_{i+2}} \ldots h_{q_m} \end{bmatrix}},$$

becomes

$$(-)^{i(i-1)} . \zeta(h_{q_1} h_{q_2} \ldots h_{q_i}) \Sigma(x-h_{q_{i+1}})(x-h_{q_{i+2}}) \ldots (x-h_{q_m})$$

$$= \zeta(h_{q_1} h_{q_2} \ldots h_{q_i}) \Sigma(x-h_{q_{i+1}})(x-h_{q_{i+2}}) \ldots (x-h_{q_m}),$$

since $(-)^{\frac{i(i-1)}{2}}=1$, which gives the well-known formulæ (enunciated by me in the London and Edinburgh Philosophical Magazine for 1839) for expressing M. Sturm's auxiliary functions in terms of the roots of the primitive, and which I therein stated were immediately deducible from the general formulæ (also enunciated in the same paper) applicable to any two functions. These more general formulæ appear to have completely escaped the notice of M. Sturm and others, who have used the special formulæ applicable to the case of one function becoming the first differential derivative of the other.

Art. (36.). In precisely the same manner, if we form as usual the ordinary syzygetic equation

$$t.f'x-\tau fx+\vartheta=0,$$

we may find the different values of t given by the complementary formulæ; and using t_i to denote the multiplier of the degree i in x, *i. e.* appertaining to the *residue* of the degree $(m-i-1)$ in x, we have

$$t_i=\Sigma\frac{\begin{bmatrix} h_{q_1}h_{q_2}\ldots h_{q_i} \\ \eta_1\ \eta_2\ \ldots\eta_{m-1} \end{bmatrix}}{\begin{bmatrix} h_{q_1}\ h_{q_2}\ \ldots h_{q_i} \\ h_{q_{i+1}}\ h_{q_{i+2}}\ldots h_{q_m} \end{bmatrix}}(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i})$$

$$=\zeta(h_{q_1}h_{q_2}\ldots h_{q_i})(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_i}).$$

Art. (37.). Thus, if we make $i=m-1$,

$$f'_1(x)=t_{m-1}=\zeta(h_{q_1}h_{q_2}\ldots h_{q_{m-1}})(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_{m-1}}).$$

It is evident from the form of $f'_1.x$ that it possesses relative to fx, the same property as fx, I mean the property that when x is indefinitely near to a real root of fx, and is passing from the inferior to the superior side of such root, $\frac{f_1x}{fx}$ like $\frac{f'(x)}{fx}$ will pass from being negative to being positive, or in other words, f_1x and $f'x$ have always the same sign in the immediate vicinity to a real root of fx. Hence it follows that $f'_1(x)$ might be used instead of $f'x$, to produce, by the Sturmian process of common measure, a series of auxiliary functions, which with fx and $f'_1.x$ would form a rhizoristic series, *i. e.* a series for determining (as in the manner of M. Sturm's ordinary auxiliaries) the number of real roots of fx comprised within given limits. The rhizoristic series generated by this process will, it is easily seen, be (to a constant factor *près*) the denominators (reckoning $+1$ as the denominator in the zero place) of the successive convergents to $\frac{f'x}{fx}$ thrown under the form of a continued fraction $\frac{1}{q_1-}\ \frac{1}{q_2-}\ldots\frac{1}{q_{n-1}-}\ \frac{1}{q_n}$; M. Sturm's own rhizoristic series, on the contrary (will be to a constant factor *près*), the denominators of the convergents to the inverse fraction $\frac{f_1x}{fx}$, which will be of the form $K.\frac{1}{q_n-}\ \frac{1}{q_{n-1}-}\ldots\frac{1}{q_2-}\ \frac{1}{q_1}$; accordingly these two

rhizoristic series will be equivalent as regards the number of changes and of combinations of sign (afforded by each) corresponding to any given value of x, of which of course the q's are linear functions. This result agrees with what has been demonstrated by me by a more general method (in the London and Edinburgh Philosophical Magazine, June and July 1853), where it has been proved, by means of a very simple theorem of determinants, that the two series

$$\frac{1}{q_1};\ \frac{1}{q_1-}\ \frac{1}{q_2-};\ \frac{1}{q_1-}\ \frac{1}{q_2-}\ \frac{1}{q_3};\ \ldots\frac{1}{q_1-}\ \frac{1}{q_2-}\ \frac{1}{q_3}\ \ldots\frac{1}{q_n}$$

and

$$\frac{1}{q_n};\ \frac{1}{q_n-}\ \frac{1}{q_{n-1}};\ \frac{1}{q_n-}\ \frac{1}{q_{n-1}}\ \frac{1}{q_{n-2}};\ \ldots\frac{1}{q_n-}\ \frac{1}{q_{n-1}-}\ \frac{1}{q_{n-2}}\ldots\frac{1}{q_1}$$

always contain (for real values of $q_1, q_2, q_3 \ldots q_n$) the same number of positive and negative signs.

Art. (38.). Having now determined the general values of $\mathfrak{S}$ and t in the equation $tf'(x)-\tau fx+\mathfrak{S}=0$ as explicit integral functions of the roots of fx, the more difficult task remains to assign to τ its value similarly expressed. This cannot readily be effected by means of substitutions in the general formulæ, the method we adopted for finding t and $\mathfrak{S}$; but all the other quantities except τ in the syzyzetic equation being integral functions of the roots, it is evident that τ also must be an integral function of the same, and to obtain it we may use the expression $\tau=\frac{tf'x-\mathfrak{S}}{fx}$.

To obtain the general form of τ by direct calculation from this formula would however be found to be impracticable; the mode I adopt therefore to discover the general expression for τ corresponding to different values of $\mathfrak{S}$, is to ascertain its value on the hypothesis of particular relations existing between the roots of fx, and then from the particular values of τ thus obtained to infer demonstratively its general form, as will be seen below. The demonstration of τ is unavoidably somewhat long, τ being in fact represented by a double sum of partial symmetrical functions.

Using the subscript indices of each function as the syzygetic equation to denote its degree in x, we have in general

$$t_{m-i-1}f'x-\tau_{m-i-2}fx+\mathfrak{S}_i=0,$$

where if we make

$$h_1-x=k_1\quad h_2-x=k_2\ldots\ldots\ldots h_m-x=k_m,$$

so that

$$h_i-h_m=k_i-k_m,$$

and therefore

$$\zeta(h_{\theta_1}\ h_{\theta_2}\ldots h_{\theta_p})=\zeta(k_{\theta_1}\ k_{\theta_2}\ldots k_{\theta_p}),$$

we have in effect found

$$\mathfrak{S}_i=\Sigma(k_{q_1}.k_{q_2}\ldots k_{q_i})\zeta(k_{q_{i+1}}\ k_{q_{i+2}}\ldots k_{q_m})$$

and

$$t_{m-i-1}=\pm\Sigma(k_{q_1}\ k_{q_2}\ldots k_{q_{m-i-1}})\zeta(k_{q_1}\ k_{q_2}\ldots k_{q_{m-i-1}});$$

we have also $f'(x)=(-)^{m-1}.\Sigma k_1\ k_2\ldots k_{m-1}$.

Let us commence with the case where $i=0$, we have then

$$\mathfrak{S}_0=\zeta(k_1\,k_2\ldots k_m)$$

$$t_{m-1}=\Sigma(k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}})\zeta(k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}}),$$

we have thus

$$(-)^m.\tau_{m-2}(k_1.k_2\ldots k_m)=-\Sigma(k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}})\times\Sigma(k_{r_1}\,k_{r_2}\ldots k_{r_{m-1}}\zeta(k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}}))+\zeta_0(k_1\,k_2\ldots k_m).$$

[It may easily be verified that the negative sign interposed between the two parts of the right-hand member of the equation has been correctly taken, for

$$\zeta(k_1\,k_2\ldots k_m\ \ ) \text{ contains a term } k_1^{2(m-1)}.k_2^{2(m-2)}\ldots k_{m-2}^2.k_{m-1}^2,$$

$$\Sigma(k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}}) \text{ contains a term } k_1\ \ .k_2\ \ \ldots k_{m-2}.k_{m-1},$$

and

$$\Sigma(k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}})\zeta(k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}}) \text{ contains a term } k_1^{2m-3}.k_2^{2m-5}\ldots k_{m-2}^3.k_{m-1}^1,$$

and thus the term $k_1^{2(m-1)}.k_2^{2(m-2)}\ldots k_{m-2}^4.k_{m-1}^2$, which does not contain $k_1\,k_2\ldots k_m$, will (as it ought to do) disappear from the right-hand side of the equation.]

Now suppose

$$k_1=k_2,$$

then

$$\zeta_0(k_1\,k_2\ldots k_m)=0,$$

and also

$$\zeta(k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}})=0,$$

except when one or the other of the two disjunctive equations

$$q_1,\ q_2,\ q_3\ldots q_{m-1}=1,\ 3,\ 4\ldots m$$

$$q_1,\ q_2,\ q_3\ldots q_{m-1}=2,\ 3,\ 4\ldots m$$

is satisfied (by a disjunctive equation, meaning an equation which affirms the equality of one set of quantities with another set the same in number, each with each, but in some unassigned order).

Hence

$$\Sigma k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}}\zeta_0(k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}})$$

$$=2k_1\,k_3\ldots k_m\zeta(k_1\,k_3\ldots k_m).$$

Hence when

$$k_1=(-)^m k_2\ \ \tau_{m-2} \text{ becomes } \frac{2}{k_1}\Sigma(k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}})\zeta_0(k_1\,k_3\ldots k_m),$$

$$i.\,e.\ 2\zeta_0(k_1\,k_3\ldots k_m)\{k_1\Sigma k_{r_3}\,k_{r_4}\ldots k_{r_{m-1}}+2k_3\,k_4\ldots k_m\},$$

the Σ referring to $r_3, r_4, \ldots, r_m$ supposed to be disjunctively equal to $3, 4, \ldots m$.

Now τ_{m-2} is of $(m-2)$ dimensions in x, and whenever more than one equality exists between the k's, $\mathfrak{S}_0$ and t_{m-1} both vanish (in fact every term in each vanishes separately), and therefore τ_{m-2}, which $=\frac{\mathfrak{S}_0+t_{m-1}f'x}{k_1.k_2\ldots k_m}$, will vanish.

Hence $(-)^m\tau_{m-2}$ must be always of the form

$$\Sigma\zeta(k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}})\times\Psi(k_{q_1}\,k_{q_2}\ldots k_{q_{m-1}};\ k_{q_m}),$$

Ψ denoting some integral function of $(m-2)$ dimensions in respect of the system of quantities $k_{q_1} k_{q_2} \ldots k_{q_m}$. The result above obtained enables us to assign the value of

$$\Psi(k_1, k_2 \ldots k_m, k_2)$$

when $k_1 = k_2$,

$$\text{viz. } k_1 \Sigma(k_{r_2} k_{r_4} \ldots k_{r_{m-1}}) + 2k_2 . k_4 \ldots k_m.$$

Now for a moment suppose, selecting $(m-1)$ terms $k_1, k_3, k_4 \ldots k_m$ out of the m terms of the k series, that

$$\Omega(k_1 k_3 k_4 \ldots k_m, k_2) = k_2^{m-2} - k_2^{m-3} . S_1(\eta_1 \eta_3 \ldots \eta_m) + k_2^{m-4} S_2(\eta_1 \eta_3 \ldots \eta_m)$$
$$\pm \&c. \mp k_2 S_{m-3}(k_1 k_3 \ldots k_m) \pm 2S_{m-2}(k_1 k_3 \ldots k_m),$$

where S_1 means that the quantities which it governs are to be simply added together, S_2 denotes that their binary, S_3 that their ternary, and in general S_r that their r-ary products are to be added together.

When $k_1 = k_2$, Ω becomes

$$k_1^{m-2} - k_1^{m-3}\big(k_1 + S_1(k_3 k_4 \ldots k_m)\big) + k_1^{m-4} . \big(k_1 S_1(k_3 k_4 \ldots k_m) + S_2(k_3 k_4 \ldots k_m)\big)$$
$$- k_1^{m-5} . \big(k_1 S_2(k_3 k_4 \ldots k_m) + S_3(k_3 k_4 \ldots k_m)\big) \pm \&c. \mp k_1 \big(k_1 S_{m-4}(k_3 k_4 \ldots k_m) + S_{m-3}(k_3 k_4 \ldots k_m)\big)$$
$$\pm 2S_{m-3}(k_3 k_4 \ldots k_m),$$

which evidently equals

$$\pm \{2S_{m-3}(k_3 k_4 \ldots k_m) + k_1 S_{m-3}(k_3 k_4 \ldots k_m)\},$$
$$i.\,e. \pm \{k_1 \Sigma(k_{r_2} k_{r_4} \ldots k_{r_{m-1}}) + 2k_3 k_4 \ldots k_m\}.$$

Hence when $k_1 = k_2$, $\Psi = \Omega$, and

$$(-)^m \tau_{m-3} = \Sigma\zeta(h_{q_1} h_{q_3} \ldots h_{q_{m-1}}) \times \Omega(k_{q_1} k_{q_3} \ldots k_{q_{m-1}} k_{q_m});$$

and so in like manner, when k_1 is equal to any one of the $(m-1)$ quantities $k_2, k_3 \ldots k_m$, the form of τ_{m-3} above written will have been correctly assumed. But τ_{m-3} may be treated as a function of $(m-2)$ dimensions in k_1, and consequently any form of $(m-2)$ dimensions in k_1, which fits it for $(m-1)$ different values of k_1, must be its general form, and accordingly we have universally,

$$(-)^m \tau_{m-3} = \Sigma\zeta(h_{q_1} h_{q_3} \ldots h_{q_{m-1}}) \times \{(x - h_{q_m})^{m-2} - (x - h_{q_m})^{m-3} S_1(x - h_{q_1}, x - h_{q_3} \ldots x - h_{q_{m-1}})$$
$$+ (x - h_{q_m})^{m-4} S_2(x - h_{q_1}, x - h_{q_3} \ldots x - h_{q_{m-1}}) \pm \&c.$$
$$\mp (x - h_{q_m}) S_{m-3}(x - h_{q_1}, x - h_{q_3} \ldots x - h_{q_{m-1}}) \pm 2S_{m-2}(x - h_{q_1}, x - h_{q_3} \ldots x - h_{q_{m-1}})\}.$$

Art. (39.). With a view to better paving our way to the general form of τ for all values of i, let us pass over the case of $i=1$ and go at once to the equation

$$t_{m-3} . f'x - \tau_{m-4} fx + \vartheta_2 = 0;$$

and to better fix our ideas let $m=7$, so that the equation becomes

$$t_4 . f'x - \tau_3 . fx + \vartheta_2 = 0;$$

we have then, preserving the same relation as before [*i. e.* using h to denote any root

of fx, and k to denote $h-x$], the equation

$$\pm k_1 k_2 k_3 k_4 k_5 k_6 k_7 . \tau_3 = \Sigma k_{q_1} . k_{q_2} \zeta(k_{q_3} k_{q_4} k_{q_5} k_{q_6} k_{q_7}) - \Sigma(k_{q_1} k_{q_2} k_{q_3} k_{q_4} k_{q_5} k_{q_6}) \times \Sigma\{(k_{q_1} k_{q_2} k_{q_3} k_{q_4}) \zeta(k_{q_5} k_{q_6} k_{q_7})\};$$

and τ_3 will vanish whenever more than three relations of equality exist between the k's, for then each term in *both* of the two sums in the right-hand member of the equation above written will separately vanish; and of course three relations of equality between the same are sufficient to make all the terms in the first of these sums vanish. This relationship between the different k's corresponding to a multiplicity 3 may arise in different ways; the multiplicity 3 may be divided into 3 units corresponding to 3 pairs of equal roots, or into 2 and 1 corresponding one set of 3 equal roots, and a second set of 2 equal roots, or may be taken "en bloc," which corresponds to the case of one set of 4 equal roots. I shall make the first of these suppositions, which will sufficiently well answer our purpose in the case before us.

Thus I shall suppose $\quad k_1 = k_4 \quad k_2 = k_5 \quad k_3 = k_6,$

then, as above remarked, $\zeta(k_{q_5} k_{q_4} k_{q_5} k_{q_6} k_{q_7}) = 0$ for all values of $q_3 q_4 q_5 q_6 q_7$, and therefore

$$\Sigma k_{q_1} k_{q_2} \zeta(k_{q_3} k_{q_4} k_{q_5} k_{q_6} k_{q_7}) = 0;$$

also $\Sigma k_{q_1} k_{q_2} k_{q_3} k_{q_4} k_{q_5} k_{q_6}$ becomes

$$k_1 k_2 k_3 (k_1 . k_2 . k_3 + 2 k_7 . k_1 k_2 + k_1 k_3 + k_2 k_3),$$

and $\zeta(\eta_{q_1} \eta_{q_2} \eta_{q_3} \eta_{q_4})$ vanishes, except for the cases where $q_1 q_2 q_3 q_4$ represent respectively, q_1 the index 1 or 4, q_2 the index 2 or 5, q_3 the index 3 or 6, and q_4 the index 7.

Hence $\quad \Sigma k_{q_1} k_{q_2} k_{q_3} k_{q_4} \zeta(k_{q_1} k_{q_2} k_{q_3} k_{q_4}) = 2^3 k_1 k_2 k_3 k_7 \zeta(k_1 k_2 k_3 k_7),$

and consequently τ_3 becomes

$$\pm 8 \zeta(k_1 k_2 k_3 k_7) \times \{k_1 k_2 k_3 + 2 k_7 (k_1 k_2 + k_1 k_3 + k_2 k_3)\}.$$

Hence we are able to predict that the general expression for our τ in the case before us will be

$$\tau_3 = \mp \Sigma\{\zeta(k_{q_1} k_{q_2} k_{q_3} k_{q_7}) \times \left\{ \begin{array}{l} (k_{q_4}^3 + k_{q_5}^3 + k_{q_6}^3) - (k_{q_4}^3 + k_{q_5}^3 + k_{q_6}^3)(k_{q_1} + k_{q_2} + k_{q_3} + k_{q_7}) \\ \quad + (k_{q_4} + k_{q_5} + k_{q_6})(k_{q_1} . k_{q_2} + k_{q_1} . k_{q_3} + k_{q_1} k_{q_7} + k_{q_2} . k_{q_3} + k_{q_4} . k_{q_7} + k_{q_5} . k_{q_7}) \\ \quad - 4(k_{q_1} k_{q_2} k_{q_3} + k_{q_1} k_{q_2} k_{q_7} + k_{q_1} k_{q_3} k_{q_7} + k_{q_2} k_{q_3} k_{q_7}) \end{array} \right\}.$$

For in the first place, the fact that the τ vanishes when more than three relations of equality exist between the k's, proves that we may assume τ_3 of the form

$$\Sigma\{\zeta(k_{q_1} k_{q_2} k_{q_3} k_{q_7}) \times \phi\{k_{q_1} k_{q_2} k_{q_3} k_{q_7}; \; k_{q_4} k_{q_5} k_{q_6}\},$$

the semicolon (;) separating the k's into two groups, in respect of each of which severally ϕ is a symmetrical form. But if in the expression last above written for τ_3 we make $\quad k_1 = k_4 \quad k_2 = k_5 \quad k_3 = k_6,$

it becomes

$$\mp 8 \zeta(k_1 k_2 k_3 k_7) \left\{ \begin{array}{l} (k_1^3 + k_2^3 + k_3^3) - (k_1^3 + k_2^3 + k_3^3)(k_1 + k_2 + k_3 + k_7) \\ \quad + (k_1 + k_2 + k_3)(k_1 k_2 + k_1 k_3 + k_2 k_3 + k_1 k_7 + k_2 k_7 + k_3 k_7) \\ \quad - 4(k_1 k_2 k_3 + k_1 k_3 k_7 + k_1 k_3 k_7 + k_2 k_3 k_7) \end{array} \right\}.$$

Now in general if

$$\sigma_r = a_1^r + a_2^r + a_3^r + \ldots + a_l^r,$$

and

$$S_r = \Sigma_1^s (a_1 . a_2 . a_3 \ldots a_r),$$

$$\sigma_r - \sigma_{r-1} S_1 + \sigma_{r-2} S_2 \pm \ldots \pm r S_r = 0.$$

Consequently the sum of the terms constituting the second factor in the above expression

$$= (3-4) k_1 . k_2 k_3 + (2-4) k_7 (k_1 k_2 + k_1 k_3 + k_2 k_3).$$

Hence the above expression becomes

$$\pm 8 \zeta (k_1 k_2 k_3 k_7) \{ k_1 . k_2 . k_3 + 2 (k_1 k_2 + k_1 k_3 + k_2 k_3) k_7 \}.$$

Thus, then, whenever $k_1 k_2 k_3$ are respectively equal to any three of the quantities $k_4 k_5 k_6 k_7$, which may take place in twenty-four different ways {twenty-four being the number of permutations of four things}, our τ_3 will have been correctly assumed; but $\zeta(k_{q_1} k_{q_2} k_{q_3} k_{q_7})$ being replaceable by $\zeta(h_{q_1} h_{q_2} h_{q_3} h_{q_7})$, the τ_3 may be treated as a cubic function in k_1, k_2, k_3, and arranged according to the powers of $k_1 k_2 k_3$ will contain only twenty terms; hence, since the assumed form is verified for more than twenty, *i. e.* for twenty-four values of h_1, h_2, h_3, it follows that the assumed form is universally identical with the form of τ, which was to be determined.

Art. (40.). Now, again, in order to facilitate the conception of the general proof, let us suppose fx to be of only five dimensions in x, i still remaining 3: it will no longer be possible when we suppose a multiplicity three to prevail among the roots, to conceive this multiplicity to be distributed into three parts, for that would require the existence of three pairs of roots, there being only five. But we may, if we please, make $h_1 = h_2 = h_3$, and $h_4 = h_5$, or else $h_1 = h_2 = h_3 = h_4$, or in any other mode conceive the multiplicity to be divided into two parts, 2 and 1 respectively, or to be taken collectively "*en bloc.*" As a mode of proceeding the more remote from that last employed, I shall choose the latter supposition. Then we obtain (τ now becoming τ_{5-2-2}, *i. e.* τ_1)

$$k_1 k_2 k_3 k_4 k_5 . \tau_1 = \pm \Sigma k_{q_1} k_{q_2} k_{q_3} k_{q_4} \times \{ \Sigma k_{q_1} k_{q_2} \zeta (k_{q_1} k_{q_2}) \},$$

and $\zeta(k_{q_1} k_{q_2})$ will vanish, except in the case where q_1 represent the indices 1 or 2 or 3 or 4, and q_2 the index 5; also

$$\Sigma k_{q_1} k_{q_2} k_{q_3} k_{q_4} = k_{q_1}^4 + 4 k_1^3 . k_5.$$

Hence our equation becomes

$$k_1^4 . k_5 . \tau = \pm (k_1^4 + 4 k_1^3 k_5) 4 k_1 . k_5 \zeta (k_1 k_5),$$

and τ becomes

$$-4 \zeta (k_1 k_5)(k_1 + 4 k_5).$$

If, now, we assume for the general value of τ in the case before us

$$\tau = \Sigma \zeta (k_{q_1} k_{q_2}) \{ (k_{q_3} + k_{q_3} + k_{q_4}) - 4 (k_{q_1} + k_{q_2}) \},$$

when $k_1 = k_2 = k_3 = k_4$, τ becomes

$$\pm 4 \zeta (k_1 k_5) \big(3 k_1 - (4 k_1 + k_5) \big),$$

i. e. $\pm 4 \zeta (k_1 k_5)(k_1 + 4 k_5)$.

Hence then for the two systems of values of $h_1\, h_2\, h_3$, viz.

$$\begin{matrix} h_1 = h_4 & h_1 = h_5 \\ h_2 = h_4 \text{ or } & h_2 = h_5 \\ h_3 = h_4 & h_3 = h_5, \end{matrix}$$

the form of τ will have been correctly assumed. But since the derived form is a linear function of h_1, h_2, h_3, this is not enough to identify the assumed with the general form, since for such verification four systems of values must be taken, four being the number of terms in a function of three variables of the first degree. If, however, we had adopted a separation of the multiplicity three into two parts, and had started with supposing $k_1 = k_2 = k_3$ $k_4 = k_5$, we should have found that τ would have become

$$= 6\zeta_0(k_1, k_5)(2k_1 + 3k_5).$$

Moreover, when these equalities subsist,

$$k_1 k_2 k_3 k_4 + k_1 k_2 k_3 k_5 + k_1 k_2 k_4 k_5 + k_1 k_3 k_4 k_5 + k_2 k_3 k_4 k_5$$

becomes $2k_1^3 k_5 + 3k_1^2 . k_5^2$, and the common factor $k_1^2 . k_4$ disappears in the course of the operations for finding τ, and eventually we have to show (in order to support the universality of the previously assumed form for τ) that

$$\eta_{q_2} + \eta_{q_3} + \eta_{q_4} - 4(\eta_{q_1} + \eta_{q_5})$$

becomes $-2\eta_{q_1} - 3\eta_5$ when

$$\eta_{q_2} = \eta_{q_3} = \eta_{q_1} = \eta_1$$

and

$$\eta_{q_4} = \eta_{q_5} = \eta_5,$$

which is evidently true. Hence then τ will have been correctly assumed for the following cases,

$$\begin{matrix} k_1 = k_2 = k_3 = k_4 \\ k_1 = k_2 = k_3 = k_4 ; \end{matrix}$$

and also for the cases

$$\left.\begin{matrix} k_1 = k_2 = k_3 \text{ and } k_4 = k_5 \\ k_1 = k_3 = k_4 \text{ and } k_2 = k_5 \\ k_2 = k_3 = k_4 \text{ and } k_1 = k_5 \end{matrix}\right\}$$

$$\left.\begin{matrix} k_1 = k_2 = k_4 \text{ and } k_3 = k_5 \\ k_1 = k_3 = k_5 \text{ and } k_2 = k_4 \\ k_2 = k_3 = k_5 \text{ and } k_1 = k_4 \end{matrix}\right\},$$

i. e. for eight cases in all, whereas four only would have sufficed. Hence, "*ex abundantiâ demonstrationis*," the form assumed for τ_1 is in the case before us the general form.

Art. (41.). We may now easily write down the general form which τ assumes for all values of i and prove its correctness. If the roots be $h_1\, h_2\, h_3 \ldots h_m$, and

$$t_{m-i-1} f'x - \tau_{m-i-2} fx + \Im_i = 0,$$

we shall have

$$\pm t_{m-i-1}=\Sigma\begin{Bmatrix}\zeta(h_{q_1}, h_{q_2}, h_{q_3}\ldots h_{q_{m-i-1}})\times[\sigma_{m-i-2}-\sigma_{m-i-3}.S_1+\sigma_{m-i-4}.S_2\mp\ \&c.\\ +(-)^{m-i-3}\sigma_1.S_{m-i-3}+(-)^{m-i-2}(\sigma_0+1)S_{m-i-2}]\end{Bmatrix},$$

where σ_r denotes in general the sum of the rth *powers* of the $(i+1)$ quantities

$$(x-h_{q_{m-i}}),\ (x-h_{q_{m-i+1}}),\ \ldots(x-h_{q_m}),$$

and S_r denotes in general the sum of the *products* of the *complementary* $(m-i-1)$ quantities

$$(x-h_{q_1}),\ (x-h_{q_2})\ldots(x-h_{q_{m-i-1}})$$

combined, r and r together. It will of course also be understood that $\sigma_0=i+1$, so that $\sigma_0+1=i+2$.

Art. (42.). To prove the correctness of this general determination of the form of r_{m-i-1}, let us suppose in general that $i+1$ relations of equality spring up between the (m) quantities $k_1, k_2, \ldots k_m$, we shall then easily obtain (N representing a certain numerical multiplier)

$$\pm Q=N.\zeta(k_1\, k_2\ldots k_{m-i-1})\frac{\Sigma k_{q_1}.k_{q_2}\ldots k_{q_{m-1}}}{k_1^{\mu_1-1}.k_2^{\mu_2-1}\ldots k_{m-i-1}^{\mu_{m-i-1}-1}},$$

$k_1, k_2\ldots k_{m-i-1}$ being what the (k) system becomes when repetitions are excluded, and being respectively supposed to occur $\mu_1, \mu_2, \ldots \mu_{m-i-1}$ times respectively, so that

$$\mu_1+\mu_2+\ldots+\mu_{m-i-1}=m;$$

the fractional part of the right-hand member of the equation immediately above written will be readily seen to be equivalent to

$$\Sigma\mu_{q_{m-i-1}}(k_{q_1}.k_{q_2}\ldots k_{q_{m-i-2}}).$$

To establish the correctness of the assumed form, we must be able, as in the particular cases previously selected, to prove two things; the one, and the more difficult thing to be proved is, that when the series of distinct quantities $k_1, k_2, k_3\ldots k_m$ become converted into μ_1 groups of k_1; μ_2 groups of $k_2\ldots\mu_{m-i-1}$ groups of k_{m-i-1}, then that

$$\Sigma\mu_{q_1}.(k_{q_1}\, k_{q_2}\, k_{q_3}\ldots k_{q_{m-i-1}}),$$

or in other terms,

$$\Sigma\pm k_{q_1}\, k_{q_2}\, k_{q_3}\ldots k_{q_{m-i-1}}\Sigma^1_{m-i-1}(\mu_q),$$

becomes identical with

$$\sigma_{m-i-2}-\sigma_{m-i-3}S_1\pm\ \&c.\ +(\sigma_0+1)S_{m-i-2}.$$

The other step to be made, and with which I shall commence, consists in showing that the number of terms in the expression last above written, considered as a function of $(m-i-2)$th degree of $(i+1)$ variables, is never greater than the entire number of ways in which $(2+1)$ quantities out of m quantities may be equated to the remaining $(m-i-1)$ quantities, viz. each of the first set respectively to all the same, or all different, or some the same and some different; in short, in any manner each of the $i+1$ quantities with some one or another (without restriction against repetitions) of the $m-i-1$ remaining quantities. This latter number being in fact the number of ways in which $(m-i-1)$ quantities may be combined $(i+1)$ together with repetitions

admissible by a well-known arithmetical theorem is $(m-i-1)^{i+1}$, and the first number is $\frac{(i+1)(i+2)\ldots(m-2}{1.2\ldots(m-i-2)}$, which is always less than the other. It remains then only to prove the remaining step of the demonstration*.

Art. (43.). To fix the ideas let $m=10$, $i=5$, and consider the expression

$$(k_5^3+k_6^3+k_7^3+k_8^3+k_9^3+k_{10}^3)-(k_5^2+k_6^2+k_7^2+k_8^2+k_9^2+k_{10}^2)(k_1+k_2+k_3+k_4)$$
$$+(k_5+k_6+k_7+k_8+k_9+k_{10})(k_1.k_2+k_1k_3+k_1k_4+k_2k_3+k_2.k_4+k_3.k_4)$$
$$-7(k_1k_2k_3+k_1k_2k_4+k_1.k_3.k_4+k_2.k_3.k_4).$$

Now suppose the six quantities k_5, k_6, k_7, k_8, k_9, k_{10} to become respectively equal each to some one or another of the four quantities k_1, k_2, k_3, k_4, as for instance, I shall suppose

$$k_5=k_6=k_7=k_1$$
$$k_8=k_9=k_2$$
$$k_{10}=k_3.$$

Then $\mu_1=4$, $\mu_2=3$, $\mu_3=2$, $\mu_4=1$,

and the formula of art. (41) becomes

$$(3k_1^3+2k_2^3+k_3^3)-(3k_1^2+2k_2^2+k_3^2)(k_1+k_2+k_3+k_4)$$
$$+(3k_1+2k_2+k_3)(k_1.k_2+k_1.k_3+k_1.k_4+k_2.k_3+k_4.k_4+k_3.k_4)$$
$$-7(k_1.k_2.k_3+k_1.k_2.k_4+k_1k_3k_4+k_2.k_3.k_4)$$
$$=3\{(k_1^3-k_1^2.\overline{(k_2+k_3+k_4+k_1)}+k_1(\overline{k_2k_3+k_2k_4+k_3}.k_4+k_4.\overline{k_3+k_3+k_4})$$
$$+2\{k_2^3-k_2^2.\overline{(k_1+k_3+k_4+k_2)}+k_2(\overline{k_1k_3+k_1k_4+k_3k_4}+\overline{k_2k_1+k_3+k_4})$$
$$+(k_3^3-k_3^2\overline{(k_1+k_2+k_4+k_3)}+k_3\overline{(k_1.k_2+k_1.k_4+k_2.k_4+k_3k_1+k_2+k_4)}$$
$$-\{k_2k_3k_4+k_1k_3k_4+k_1k_2k_4+k_1k_2k_3\}$$
$$=-k_1k_2k_3-2k_1k_2k_4-3k_1k_3k_4-4k_2k_3k_4$$
$$=-k_1k_3k_3k_4\left\{\frac{\mu_1}{k_1}+\frac{\mu_2}{k_2}+\frac{\mu_3}{k_3}+\frac{\mu_4}{k_4}\right\}.$$

In the above investigation the quantities which with their repetitions make up the k's system, are k_4, k_1, k_2, k_3, *appearing* respectively 1, 2, 3, 4 times, that is to say *repeated* 0, 1, 2, 3 times; 7 is 1 more than the sum of the repetitions $0+1+2+3$, and the numbers 1, 2, 3, 4 arise from subtracting from 7 the sums $1+2+3$; $0+2+3$; $0+1+3$; $0+1+2$; respectively, so that the remainders 1, 2, 3, 4 denote respectively one more than the number of *repetitions* of k_4, k_1, k_2, k_3, *i. e.* are the number of *appear-*

* If this first step of the demonstration appear unsatisfactory or subject to doubt, it may be dispensed with, and the result obtained in the succeeding article (the demonstration of which is wholly unexceptionable) being assumed, it may be proved that the formula there obtained on a particular hypothesis must be universally true, in precisely the same way and by aid of the same Lemma in and by aid of which the formula obtained in the Supplement to this section for the simplified quotients to $\frac{f'x}{fx}$ upon a like particular hypothesis is shown to be of universal application, *i. e.* by showing that otherwise a function of $2i-1$ variables would contain a function of $2i$ variables as a factor.

ances of k_0, k_1, k_2, k_3; and thus with a slight degree of attention to the preceding process the reader may easily satisfy himself that the preceding demonstration (although not so expressed) is in essence universal, and the form of r as an explicit function of x and of the roots of $f(x)$ is thus completely established for all values of m and of i.

Supplement to SECTION III.

On the Quotients resulting from the process of continuous division ordinarily applied to two Algebraical Functions in order to determine their greatest Common Measure.

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Art. (*a*.)* We have now succeeded in exhibiting the forms of the numerators and denominators of $\frac{f'x}{fx}$ developed into a continued fraction in terms of the differences of the roots and factors of fx. It remains to exhibit the quotients themselves of this continued fraction under a similar form.

Lemma.—*An equation being supposed of an arbitrary degree* n, *there exists no function of* n *and of less than* 2i *of the coefficients*†, *which vanishes for all values of* n *whenever the* n *roots reduce in any manner to* i *distinct groups of equal roots; or in other words, any function of* n *and the first* 2i−1 *coefficients of an equation of the* n*th degree, which vanishes for all values of* n *in every case where the roots retain only* i *distinct names, must be identically zero.*

To render the statement of the proof more simple, let i be taken equal to 3. And let the roots be supposed to reduce to p roots a, q roots b, and r roots c. And let s_r in general denote the sum of the rth powers of the roots. Then we have evidently

$$\begin{aligned}
p + q + r &= s_0\\
pa + qb + rc &= s_1\\
pa^2 + qb^2 + rc^2 &= s_2\\
pa^3 + qb^3 + rc^3 &= s_3\\
pa^4 + qb^4 + rc^4 &= s_4,\\
&\text{\&c. \&c., } ad\ infinitum.
\end{aligned}$$

Eliminating p, q, r between the first, second, third and fourth equations, we obtain

$$\begin{vmatrix} 1 & 1 & 1 & s_0 \\ a & b & c & s_1 \\ a^2 & b^2 & c^2 & s_2 \\ a^3 & b^3 & c^3 & s_3 \end{vmatrix} = 0.$$

* The articles in this and subsequent sections to which Latin or Greek letters are prefixed, although in strict connexion with the context, are supplementary in the sense of having been supplied since the date when the paper was presented for reading to the Royal Society. All the articles marked with numbers (from 1 to 72), and the Introduction, appeared in the memoir as originally presented to the Society, June 16, 1853.

† In the proposition thus enunciated the coefficient of the highest power of x is supposed to be a numerical quantity.

In like manner eliminating ap, bq, cr between the second, third, fourth and fifth equations, we have

$$\begin{vmatrix} 1 & 1 & 1 & s_1 \\ a & b & c & s_2 \\ a^2 & b^2 & c^2 & s_3 \\ a^3 & b^3 & c^3 & s_4 \end{vmatrix} = 0;$$

and so in general we have for all values of e,

$$\begin{vmatrix} 1 & 1 & 1 & s_e \\ a & b & c & s_{e+1} \\ a^2 & b^2 & c^2 & s_{e+2} \\ a^3 & b^3 & c^3 & s_{e+3} \end{vmatrix} = 0;$$

whence it may immediately be deduced, that, upon the given supposition of there being only three groups of distinct roots, we must have the following infinite system of coexisting equations satisfied, viz.—

$$\begin{aligned} s_0t+s_1u+s_2v+s_3w=0 \quad & \text{say } L_0=0 \\ s_1t+s_2u+s_3v+s_4w=0 \quad & L_1=0 \\ s_2t+s_3u+s_4v+s_5w=0 \quad & L_2=0 \\ s_3t+s_4u+s_5v+s_6w=0 \quad & L_3=0 \\ s_4t+s_5u+s_6v+s_7w=0, \quad & L_4=0, \\ \text{\&c. \&c. \&c.} \quad & \text{\&c.,} \end{aligned}$$

and conversely, when this infinite system of equations is satisfied the roots must reduce themselves to three groups of equal roots.

Let now φ be any function of $s_0\ s_1\ s_2 \ldots s_{\infty}$ which vanishes when this is the case. Then φ must necessarily contain as a factor some derivee of the infinite system of equations above written, *i. e.* some function of $s_0\ s_1\ s_2$, &c., which vanishes when these equations are satisfied; *i. e.* some conjunctive of the quantities $L_0\ L_1\ L_2\ L_3$; but it is obviously impossible in any such conjunctive to exclude s_0 from appearing, unless by introducing some other s with an index higher than e, and consequently φ cannot be merely a function of $s_0\ s_1\ s_2\ s_3\ s_4\ s_5$, nor consequently of n, and the first five coefficients; or if such, it is identically zero, and so in general any function of n, and only $2i-1$ of the coefficients, which vanishes when the roots reduce to i groups of equal roots, must be identically zero, as was to be proved.

Art. (*b.*) It ought to be observed that the preceding reasoning depends essentially upon the circumstance of n being left arbitrary. If n were given the proposition would no longer be true. In fact, on that supposition, the n roots reducing to i distinct roots would imply the existence of $n-i$ conditions between the n roots; and consequently $n-i$ independent equations would subsist between the n coefficients, and functions could be formed of i only of the coefficients, which would satisfy the prescribed condition of vanishing when the roots resolved themselves into i groups of distinct identities.

Art. (*c.*) Let $D_{r_1 r_2 \ldots r_i}$ be used in general to denote the determinant

$$\begin{matrix} s_{r_1} & s_{r_1+1} & s_{r_1+2} & \ldots & s_{r_1+i-1} \\ s_{r_2} & s_{r_2+1} & s_{r_2+2} & \ldots & s_{r_2+i-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ s_{r_i} & s_{r_i+1} & s_{r_i+2} & \ldots & s_{r_i+i-1} \end{matrix}$$

then the simplified ith Sturmian residue R_i may be expressed under the form

$$D_{1,2,3\ldots i}x^{n-i-1} - D_{2,3,\ldots i+1}x^{n-i-2} + D_{3,4\ldots(i+2)}x^{n-i-3} \ldots \pm D_{n-i,n-i-1,\ldots n}$$

which is easily identifiable with the known expression for such residue.

Now obviously the necessary and sufficient conditions in order that the n roots may consist of only repetitions of i distinct roots is, that R_i shall be identically zero, that is to say, we must have

$$D_{1,2,\ldots i}=0 \quad D_{2,3\ldots i+1}=0 \ldots\ldots D_{n-i,n-i-1,\ldots n}=0.$$

But the reasoning of the preceding article shows that although these equations are necessary and sufficient, they are but a selected system of equations of an infinite number of similar equations which subsist*, and that, in fact, whatever be the value of (n), we may take $r_1\ r_2 \ldots r_i$ perfectly arbitrary and as great as we please, and the equation

$$D_{r_1 r_2 \ldots r_i}=0$$

must exist by virtue of the existence of the $n-i$ equations last above written.

Art. (*d.*) I now return to the question of expressing the successive quotients of $\frac{f'x}{fx}$ as functions of the differences of the roots and factors; that they must be capable of being so expressed is an obvious consequence of the fact, that the numerators and denominators of the convergents have been put under that form, since if

$$\frac{N_{i-2}}{D_{i-2}}, \quad \frac{N_{i-1}}{D_{i-1}}, \quad \frac{N_i}{D_i}$$

are any three consecutive convergents of the continued fraction

$$\frac{1}{Q_1-} \quad \frac{1}{Q_2-} \ldots\ldots \frac{1}{Q_i},$$

we must have

$$D_{i-2}.N_i - N_{i-2}.D_i = Q_i.$$

It would not, however, be easy to perform the multiplications indicated in the above equation, so as to obtain Q_i under its reduced form as a linear function of x. I proceed therefore to find Q_i constructively in the following manner.

Let R_{i-2}, R_{i-1}, R_i be three consecutive residues, $f'x$ counting as the residue in the zero place, then $Q_i = \frac{R_{i-2}-R_i}{R_{i-1}}$ and is of the form $\frac{p}{q}x + \frac{p'}{q'}$.

* But *quære* whether any other *sufficient* system can be found of equations so few in number as this system.

Now in general if we call the n roots of fx, where the coefficient of x^n is supposed to be unity, $h_1\ h_2 \ldots h_n$, and if we use Z_i to denote $\Sigma\zeta(h_{\theta_1}\ h_{\theta_2} \ldots h_{\theta_i})$*, with the convention that $Z_i = n$, $Z_0 = 1$, we have, employing (i) to denote $\frac{1}{2}\{(-1)^i + 1\}$,

$$R_i = \frac{Z_{i-1}^2 . Z_{i-3}^2 \ldots Z_{(i)}^2}{Z_{i-1}^2 . Z_{i-3}^2 \ldots Z_{(i)+1}^2} . \Sigma\{\zeta(h_{\theta_1}\ h_{\theta_2} \ldots h_{\theta_{i+1}})(x - h_{\theta_{i+2}})(x - h_{\theta_{i+3}}) \ldots (x - h_{\theta_n})\}$$

$$R_{i-1} = \frac{Z_{i-2}^2 . Z_{i-4}^2 \ldots Z_{(i)+1}^2}{Z_{i-1}^2 . Z_{i-3}^2 \ldots Z_{(i)}^2} \Sigma\{\zeta(h_{\theta_1}\ h_{\theta_2} \ldots h_{\theta_i})(x - h_{\theta_{i+1}})(x - h_{\theta_{i+2}}) \ldots (x - h_{\theta_n})\}$$

$$R_{i-2} = \frac{Z_{i-3}^2 . Z_{i-5}^2 \ldots Z_{(i)}^2}{Z_{i-2}^2 . Z_{i-4}^2 \ldots Z_{(i)+1}^2} \Sigma\{\zeta(h_{\theta_1}\ h_{\theta_2} \ldots h_{\theta_{i-1}})(x - h_{\theta_i})(x - h_{\theta_{i+1}}) \ldots (x - h_{\theta_n})\}.$$

The part of R_{i-1} within the sign of summation is

$$Z_i x^{n-i} - \Sigma(h_{\theta_{i+1}} + h_{\theta_{i+2}} + \ldots + h_{\theta_n})\zeta(h_{\theta_1}\ h_{\theta_2} \ldots h_{\theta_i}) x^{n-i-1} + \&c.,$$

say $$Z_i x^{n-i} - Z_i' x^{n-i-1} + \&c.,$$

and the part of R_{i-2} within the sign of summation is

$$Z_{i-1} x^{n-i+1} - Z_{i-1}' x^{n-i} + \&c.,$$

and $$Z_i^2 . \frac{Z_{i-1} x^{n-i+1} - Z_{i-1}' x^{n-i}}{Z_i x^{n-2} - Z_i' x^{n-i-1}} = Z_{i-1} Z_i x + (Z_{i-1} . Z_i' - Z_i Z_{i-1}') + \text{an algebraic fraction.}$$

Hence
$$Q_i = \frac{1}{Z_i^2} . \frac{Z_{i-3}^2 . Z_{i-5}^2 \ldots Z_{(i)}^2}{Z_{i-2}^2 . Z_{i-3}^2 \ldots Z_{(i)+1}^2} . \left\{\frac{Z_{i-3}^2 . Z_{i-4}^2 \ldots Z_{(i)+1}^2}{Z_{i-1}^2 . Z_{i-3}^2 \ldots Z_{(i)}^2}\right\}^{-1}$$
$$\times \{Z_{i-1} . Z_i x + (Z_{i-1} Z_i' - Z_i Z_{i-1}')\}$$
$$= \frac{Z_{i-1}^2}{Z_i^2} . \frac{Z_{i-3}^4 . Z_{i-5}^4 \ldots Z_{(i)}^4}{Z_{i-2}^4 . Z_{i-4}^4 \ldots Z_{(i)+1}^4} . T_i$$

T_i denoting $Z_{i-1} . Z_i x + (Z_{i-1} Z_i' - Z_i . Z_{i-1}')$.

Art. (e.) If the process of obtaining the successive quotients and residues be considered, it will easily be seen that each step in the process imports two new coefficients into the quotients, the first quotient containing no literal quotient in the part multiplying x and containing the first literal coefficient in the other part, the second quotient containing two literal coefficients in the one part and three in the other, and in general the ith quotient containing $2i-2$ of the letters in the one part and $2i-1$ of them in the other. Hence T_i being made equal to $L_i . x + M_i$, L_i contains $2i-2$ and M_i contains $2i-1$ of the literal coefficients of fx.

Moreover, we have

$$Z_i \text{ of the form } T_i^2 . \frac{P_{i-2} - mP_i}{P_{i-1}},$$

where

$$P_{i-1} = \Sigma\zeta(h_{\theta_1}\ h_{\theta_2} \ldots h_{\theta_i}) \eta_{\theta_{i+1}}\ \eta_{\theta_{i+2}} \ldots \eta_{\theta_n}$$

$$P_{i-2} = \Sigma\zeta(h_{\theta_1}\ h_{\theta_2} \ldots h_{\theta_{i-1}}) \eta_{\theta_i}\ \eta_{\theta_{i+1}} \ldots \eta_{\theta_n},$$

* ζ it will be remembered is the symbol of the operation of taking the product of the squares of the differences of the quantities which it governs.

and P_i, which is the ith simplified residue, vanishes when the n roots in any manner become reduced to only i distinct groups.

I proceed to show that if we make

$$A_i x+B_i=U_i=A^2_{i,1}(x-h_1)+A^2_{i,2}(x-h_2)+\ldots+A^2_{i,n}(x-h_n),$$

where in general

$$A_{i,s} \text{ represents } \Sigma\zeta_\theta(h_{\theta_1} h_{\theta_2}\ldots h_{\theta_{i-1}})(h_s-h_{\theta_1})(h_s-h_{\theta_2})\ldots(h_s-h_{\theta_{i-1}}),$$

then will

$$T_i=U_i.$$

It will be observed that $A_{i,s}$ is identical with what the simplified denominator of the $(i-1)$th convergent becomes when we write h_s in place of x, and consequently, when arranged according to the powers of h_s, will be of the form

$$c_1.h_s^{i-1}+c_2 h_s^{i-2}+\ldots+c_i$$

where c_1, c_2, ... c_i are functions of the coefficients, but containing no more of them than enters into Q_{i-1}, *i. e.* containing only $2i-2$ of them.

Now A_i is made up of terms, each consisting of some binary product of

$$c_1,\ c_2,\ \ldots,\ c_i$$

combined with some term of the series

$$\Sigma h^{2i-2},\ \Sigma h^{2i-3}\ \ldots\ \Sigma h^0;$$

and any one of this latter set of terms expressed as a function of the coefficients of fx contains at most $2i-2$ of them.

Hence only $2i-2$ of the coefficients enter into A_i, and in like manner only $2i-1$ of them into B_i.

The number of letters, therefore, in A_i and in B_i is the same as in L_i and in M_i, viz. $2i-2$ and $2i-1$ respectively.

Now let the roots consist of only i distinct groups of equal roots, so that T_i becomes $=Z_i^2.\frac{P_{i-2}}{P_{i-1}}$.

I shall show that in whatever way the equal roots are supposed to be grouped upon this supposition, there will result the equation

$$T_i=U_i,$$

where

$$T_i=\{\Sigma\zeta_\theta(\eta_{\theta_1}\eta_{\theta_2}\ldots\eta_{\theta_i})\}^2.\frac{P_{i-2}}{P_{i-1}}$$

$$P_{i-2}=\Sigma\{\eta_{\theta_1}\eta_{\theta_{i+1}}\ldots\eta_{\theta_m}\zeta_\theta(\eta_{\theta_1}\eta_{\theta_2}\ldots\eta_{\theta_{i-1}})\}$$

$$P_{i-1}=\Sigma\{\eta_{\theta_{i+1}}\eta_{\theta_{i+2}}\ldots\eta_{\theta_m}\zeta_\theta(\eta_{\theta_1}\eta_{\theta_2}\ldots\eta_{\theta_i})\},$$

and

$$H_i=A^2_1.\eta_1+A^2_2.\eta_2+\ldots+A^2_n.\eta_n,$$

A_s meaning

$$\Sigma\{(\eta_s-\eta_{\theta_1})(\eta_s-\eta_{\theta_2})\ldots(\eta_s-\eta_{\theta_{i-1}})\zeta_\theta(\eta_{\theta_1}\eta_{\theta_2}\ldots\eta_{\theta_{i-1}})\},$$

and η_ω meaning $x-h_\omega$.

Let the n factors be constituted of m_1 factors η_1 m_2 factors $\eta_2 \ldots m_i$ factors η_i. Then

$$Z_i = \mu^2 \zeta(\eta_1 \eta_2 \ldots \eta_i),$$

where
$$\mu = m_1 . m_2 \ldots m_i$$
$$P_{i-1} = \mu^2 \zeta(\eta_1 \eta_2 \ldots \eta_i) \eta_1^{m_1-1} . \eta_2^{m_2-1} \ldots \eta_i^{m_i-1},$$

and
$$P_{i-2} = \mu_1 . \zeta_2(\eta_2 \eta_3 \ldots \eta_i) \eta_1^{m_1} . \eta_2^{m_2-1} \ldots \eta_i^{m_i-1}$$
$$+ \mu_2 . \zeta_2(\eta_1 \eta_3 \ldots \eta_i) \eta_1^{m_1-1} . \eta_2^{m_2} \ldots \eta_i^{m_i-1}$$
$$+ \text{ \&c. \&c.}$$
$$+ \mu_i \zeta_2(\eta_1 \eta_2 \ldots \eta_{i-1}) \eta_1^{m_1-1} . \eta_2^{m_2-1} \ldots \eta_i^{m_i},$$

where
$$\mu_1 = \frac{\mu}{m_1}, \ \mu_2 = \frac{\mu}{m_2} \ldots\ldots \mu_i = \frac{\mu}{m_i}.$$

Hence
$$T_i = \mu^2 \zeta(\eta_1 \eta_2 \ldots \eta_i) \left\{ \frac{\eta_1 \zeta(\eta_2 \eta_3 \ldots \eta_i)}{m_1} + \frac{\eta_2 \zeta(\eta_1 \eta_3 \ldots \eta_i)}{m_2} + \ldots + \frac{\eta_i \zeta(\eta_1 \eta_2 \ldots \eta_{i-1})}{m_i} \right\}.$$

Again, in U_i the term containing η_1 will be

$$m_1 \eta_1 \Sigma\{(\eta_1 - \eta_2)(\eta_1 - \eta_3) \ldots (\eta_1 - \eta_i) \zeta(\eta_2 \eta_3 \ldots \eta_i)\}^2$$
$$= m_1 \eta_1 \times (m_2 . m_3 \ldots m_i)^2 \times (\eta_1 - \eta_2)^2 (\eta_1 - \eta_3)^2 \ldots (\eta_1 - \eta_i)^2 \{\zeta \eta_2 . \eta_3 \ldots \eta_i\}^2$$
$$= \frac{\mu^2}{m_1} \eta_1 \times \zeta(\eta_1 \eta_2 \ldots \eta_i) \zeta(\eta_2 \eta_3 \ldots \eta_i).$$

Hence
$$U_i = \mu^2 \zeta(\eta_1 \eta_2 \ldots \eta_i) \left\{ \frac{\eta_1 \zeta \eta_2 \eta_3 \ldots \eta_i}{m_1} + \frac{\eta_2 \zeta \eta_1 \eta_3 \ldots \eta_i}{m_2} + \text{ \&c.} \right\} = T_i.$$

Hence, therefore, $U_i - T_i$ vanishes whenever the roots of fx contain only i distinct groups of equal roots, and it has been shown that U_i and T_i each contain only $2i-1$ of the coefficients of fx, so that $U_i - T_i$ is a function only of n and these $2i-1$ letters, and consequently by virtue of the Lemma in Art. (*a*.) $U_i - T_i$ is universally zero, *i. e.* U_i is identical with T_i, as was to be proved. In the same manner as observed in a preceding marginal note, the expression given in the antecedent articles for the numerator of the ith convergents having been verified for the case of the roots consisting of only i distinct groups, could have been at once inferred to be generally true by aid of the Lemma above quoted.

Art. (*f*.) Since the coefficient of x in T_i is $Z_{i-1} \times Z_1$, we deduce the unexpected relation

$$\Sigma \zeta(h_1 h_2 \ldots h_{i-1}) \times \Sigma \zeta(h_1 h_2 \ldots h_i) = P_1^2 + P_2^2 + \ldots + P_n^2,$$

where
$$P_s = \Sigma\{(h_s - h_{\theta_1})(h_s - h_{\theta_2}) \ldots (h_s - h_{\theta_{i-1}}) \zeta(h_{\theta_1} h_{\theta_2} \ldots h_{\theta_{i-1}})\}.$$

So that every simplified Sturmian quotient to $\frac{f'x}{fx}$, when the (n) roots of fx are real, will be the sum of n squares. But the equation is otherwise remarkable, in exhibiting the product of the sum of $\frac{n.(n-1)\ldots(n-i+2)}{1.2\ldots(i-1)}$ squares by another sum of $\frac{n(n-1)\ldots(n-i-1)}{1.2\ldots i}$ squares under the form of the sum of n squares.

If we call the ith simplified denominator to the Sturmian convergents to $\frac{f'x}{fx}$, $D_i(x)$, and if we call the ith simplified quotient $X_i(x)$, we have

$$X_i x = \Sigma_n^1 (D_{i-1} h_s)^2 . (x - h_s).$$

If we construct the numerators and denominators of the convergents to

$$\frac{1}{Q_1} - \frac{1}{Q_2} - \frac{1}{Q_3} \cdots \frac{1}{Q_i}$$

according to the general rule for continued fractions as functions of Q_1, Q_2, Q_3, &c., so that calling the denominators Δ_1, Δ_2, Δ_3, &c. Δ_i,

$$\Delta_1 = Q, \ \Delta_2 = Q_1 Q_2 - 1 \ldots\ldots \Delta_i = Q_i \Delta_{i-1} - \Delta_{i-2},$$

we have

$$\Delta_{i-1} x = \frac{Z_{i-2}^2 . Z_{i-4}^3 \ldots Z_{(i-1)}}{Z_{i-1}^2 . Z_{i-3}^2 \ldots Z_{(i)}^2} D_{i-1}(x),$$

$\Delta_{i-1} . x$ being in fact the multiplier of $f'x$ in the equation which connects fx and $f'x$ with the $i-1$th *complete* residue, and consequently retaining $Q(x)$ to designate the complete ith quotient, we have

$$Q_i(x) = \frac{Z_{i-1}^2}{Z_i^2} \cdot \frac{Z_{i-3}^1 . Z_{i-5}^4 \ldots Z_{(i)}^4}{Z_{i-2}^4 . Z_{i-4}^4 \ldots Z_{(i)+1}^4} \Sigma \{D_{i-1} . h_s\}^2 (x - h_s)$$

$$= \frac{Z_{i-1}^5}{Z_i^3} \cdot \frac{Z_{i-3}^6 . Z_{i-5}^6 \ldots Z_{(i)}^6}{Z_{i-2}^5 . Z_{i-4}^5 \ldots Z_{(i)+1}^5} \Sigma \left\{ \{\Delta_{i+1}, h_s\}^2 (x - h_s) \right\},$$

which equation gives the connexion between the form of any quotient and that of the immediately preceding convergent denominator of the continued fraction which expresses $\frac{f'x}{fx}$.

Art. (*g*.) I have found that the coefficients of the n factors of fx in the expression above given for the quotients possess the property that the sum of their square roots taken with the proper signs is zero for each quotient except the first (the coefficients for the first being all units), *i. e.* $D_i h_1 + D_i h_2 + \ldots D_i h_n = 0$ for all values of i except $i = 1$. Moreover I find that the determinant formed by the n sets of the n coefficients of the factors of fx in the complete set of n quotients is identically zero, *i. e.* the Determinant represented by the square matrix

$$\left\{ \begin{array}{cccc} 1 & 1 & 1 & 1 \\ (D_1 h_1)^2 & (D_1 h_2)^2 & (D_1 h_3)^2 & \ldots (D_1 h_n)^2 \\ (D_2 . h_1)^2 & (D_2 h_2)^2 & (D_2 h_3)^2 & \ldots (D_2 h_n)^2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (D_{n-1} h_1)^2 & (D_{n-1} h_2)^2 & (D_{n-1} h_3)^2 & \ldots (D_{n-1} h_n)^2 \end{array} \right\} = 0.$$

Art. (*h*.) It should be observed that U_i is the form of the simplified quotients for all the quotients except the nth (*i. e.* the last), for which the simplified form is not U_n, but $U_n \div \zeta(h_1 h_2 \ldots h_n)$, which arises from the circumstance of the last divisor, which is the final Sturmian residue, not containing x; it being evidently the case that the division

of a rational function of x by another one degree lower, introduces into the integral part of the quotient the square of the leading coefficient of the divisor, *subject to the exception* that when the divisor is of the degree zero, the simple power enters in lieu of the square. The general formula gives for the reduced nth quotient the expression

$$\Sigma\{\overline{(h_1-h_2}\,\overline{h_1-h_3}\ldots\overline{h_1-h_n}\zeta_0(h_2\,h_3\ldots h_n))^2(x-h_1)\},$$

which equals

$$\zeta(h_1\,h_2\ldots h_n)\Sigma\zeta(h_2\,h_3\ldots h_n)(x-h_1).$$

Rejecting the first factor, we have

$$\Sigma\zeta_0(h_2\,h_3\ldots h_n)(x-h_1),$$

which is equal to the penultimate residue, which residue is (as it evidently ought to be) identical with the simplified last quotient.

Art. (*i.*) We have thus succeeded in giving a perfect representation of $\frac{f'x}{fx}$, *i. e.* of

$$\frac{1}{x-h_1}+\frac{1}{x-h_2}+\ldots\ldots+\frac{1}{x-h_n},$$

under the form of a continued fraction of the form

$$\frac{1}{m_1(x-e_1)-}\quad\frac{1}{m_2(x-e_2)-}\quad\ldots\ldots\quad\frac{1}{m_n(x-e_n)},$$

where $m_1\,m_2\ldots m_n$; $e_1\,e_2\ldots e_n$ are all determinate and known functions of $h_1\,h_2\ldots h_n$.

We may by means of this identity, differentiating any number of times with respect to x both sides of the equation, obtain analogous expressions for the series

$$\frac{1}{(x-h_1)^i}+\frac{1}{(x-h_2)^i}+\ldots+\frac{1}{(x-h_n)^i}.$$

But to do this we must be in possession of a rule for the differentiation of continued fractions whose quotients are linear functions of the variable. I subjoin here the first step only toward such investigation.

Let the denominator of

$$\frac{1}{q_1-}\,\frac{1}{q_2-}\ldots\ldots\frac{1}{q_n}$$

where $q_1\,q_2\ldots q_n$ are any n arbitrary quantities, be denoted by $[q_1, q_2, q_3\ldots q_n]$, so that the entire fraction will be equal to

$$\frac{[q_2\,q_3\ldots q_n]}{[q_1\,q_2\,q_3\ldots q_n]},$$

any such quantity as $[q_i\,q_{i+1}\ldots q_n]$ may be termed a Cumulant, of which $q_i, q_{i+1}\ldots q_n$ may be severally termed the elements or Components, and the complete arrangement of the elements may be termed the Type. The cumulant corresponding to any Type remains unaffected by the order of the elements in the Type being reversed, as is evident from any cumulant being in fact representable under the form of a symmetrical determinant,

thus *ex. gr.* the cumulant $[q_1 q_2 q_3 q_4]$ may be represented by the determinant

$$\begin{matrix} q_1 & 1 & 0 & 0 \\ 1 & q_2 & 1 & 0 \\ 0 & 1 & q_3 & 1 \\ 0 & 0 & 1 & q_4, \end{matrix}$$

and $q_4 q_3 q_2 q_1$ will in like manner be represented by the determinant

$$\begin{matrix} q_4 & 1 & 0 & 0 \\ 1 & q_3 & 1 & 0 \\ 0 & 1 & q_2 & 1 \\ 0 & 0 & 1 & q_1, \end{matrix}$$

which is equal to the former.

Art. (j.) Let it be proposed in general to find the first differential coefficient in respect to x of the fraction

$$\frac{[q_i\, q_{i+1} \ldots . q_n]}{[q_1\, q_2\, q_3 \ldots q_n]} = \mathrm{F}_i,$$

where each q is a function of one or more variables.

I find that the variation of F_i may be expressed as follows:—

$$-\delta \mathrm{F}_i = \{\delta[q_1, q_2 \ldots q_{i-2}, q_n] + \delta[q_1, q_2 \ldots q_{i-2}, q_{n-1}] \cdot q_n^2$$

$$+\delta[q_1, q_2, q_3 \ldots q_{i-2}, q_{n-2}] \cdot [q_n, q_{n-1}]^2 + \&c. + \delta[q_1, q_2, q_3 \ldots q_{i-2}, q_{i-1}] \cdot [q_n, q_{n-1}, q_{n-2} \ldots q_i]^2\}$$

$$\div [q_1, q_2, q_3 \ldots q_n]^2.$$

Art. (k.) Suppose $i=2$, and $q_1 = a_1 x + b_1$, $q_2 = a_2 x + b_2 \ldots\ldots q_n = a_n x + b_n$, we shall have by virtue of the above equation,

$$\frac{d}{dx} \mathrm{F}_2, \textit{ i. e. } \frac{d}{dx} \cdot \left\{ \frac{1}{q_1} - \frac{1}{q_2} - \frac{1}{q_3} \ldots \frac{1}{q_n} \right\}$$

$$= -\frac{1}{[q_1 q_2 \ldots q_n]^2} \{a_n \cdot 1^2 + a_{n-1} \cdot q_n^2 + a_{n-2} \cdot [q_n, q_{n-1}]^2 + \&c. + a_1 [q_n, q_{n-1}, q_{n-2} \ldots q_2]^2\}.$$

If we call $\mathrm{F}_2 = \frac{\phi x}{fx}$ every such quantity as $[q_n, q_{n-1} \ldots q_i]$ represents to a constant factor *près* the $(i-1)$th simplified residue (ϕx counting as the first of them) to $\frac{\phi x}{fx}$, and making certain obvious but somewhat tedious reductions, and rejecting the common factor $-\frac{1}{(fx^2)}$, we obtain the expression

$$\frac{\mathrm{C}_0 \cdot \mathrm{R}_1^2}{\mathrm{C}_1} + \frac{\mathrm{R}_2^2}{\mathrm{C}_1 \cdot \mathrm{C}_2} + \frac{\mathrm{R}_3^2}{\mathrm{C}_2 \cdot \mathrm{C}_3} + \ldots\ldots + \frac{\mathrm{R}_n^2}{\mathrm{C}_{n-1} \cdot \mathrm{C}_n} = (\phi x \cdot f'x - \phi' x f x),$$

where $\mathrm{R}_1\ \mathrm{R}_2 \ldots \mathrm{R}_n$ represent ϕx and the successive simplified residues to fx, ϕx, and

C_i means the coefficient of the highest power of x in R_i, and C_0 the first coefficient in fx*.

Art. (*l.*) If we take $g(x)$ of the same degree as $f(x)$ and for greater simplicity make the first coefficients in $f(x)$ and $g(x)$, each of them unity, the successive simplified residues to $\frac{gx}{fx}$ will be identical with the simplified residues to $\frac{-fx+gx}{gx}$ (including amongst them the quantity $gx-fx$ itself), and since

$$(fx-g(x))g'x-(fx-g(x))'gx=(g'xfx-f'x.gx),$$

the right-hand side of the equation above written, when the residues are made to refer to f and g, instead of referring to f and φ, are taken of the same degree in x, becomes equal to $f'xgx-fxg'x$; and if we now agree to consider f and g as homogeneous functions each of the nth degree in x and 1, the equation becomes

$$\frac{R_1^2}{C_1}+\frac{R_2^2}{C_1.C_2}+\frac{R_3^2}{C_2.C_3}+\cdots\cdots+\frac{R_n^2}{C_{n-1}.C_n}$$

$$=\{f(x,1)\frac{d}{dx}g(x,1)-g(x,1)\frac{d}{dx}f(x,1)\}=\frac{1}{n}\left(x\frac{d}{dx}f+\frac{d}{d1}f\right)\left(\frac{d}{dx}g\right)-\frac{1}{n}\left(x\frac{d}{dx}g+\frac{d}{d1}.g\right)\left(\frac{d}{dx}f\right)$$

$$=\frac{1}{n}\left\{\frac{df}{d1}.\frac{dg}{dx}-\frac{df}{dx}.\frac{dg}{d1}\right\}=\frac{1}{n}\mathrm{J}(f,g),$$

where J indicates the Jacobian of the given functions f and g in respect to the variables x and 1, meaning thereby the so-called *Functional Determinant* of JACOBI to f and g in respect of x and 1, which equation also obviously must continue to hold good when we restore to the coefficients of x^n in f and g their general values.

It may happen that for particular relations between the coefficients of f and g

* This result may be obtained directly as follows:—

Let fx, φx and the $(m-1)$ complete Sturmian residues be called $\rho_0\ \rho_1\ \rho_2\ldots\rho_n$; let the n complete quotients be called $q_1\ q_2\ldots q_n$, and let the allotrious factors to the residues $\rho_2,\ \rho_3,\ \ldots\ \rho_n$ be called $\mu_2\ \mu_3\ldots\mu_n$; then

$$\rho_0=q_1.\rho_1-\rho_2;\ \rho_1=q_2\ \rho_2-\rho_3;\ \rho_2=q_3\ \rho_3-\rho_4;\ \&c.$$

hence
$$\rho_1\delta\rho_0-\rho_0\delta\rho_1=\rho_1^2\delta q_1+(\rho_2\delta\rho_1-\rho_0\delta\rho_2)$$
$$=\rho_1^2\delta q_1+\rho_2^2\delta q_2+(\rho_3\delta\rho_2-\rho_2\delta\rho_3)$$
$$=\&c.$$
$$=\rho_1^2\delta q_1+\rho_2^2\delta q_2+\rho_3^2\delta q_3+\ldots+\rho_n^2\delta q_n;$$

but we have in general $\rho_i=\mu_i.R_i$;

hence
$$\delta q_i=\frac{C_{i-1}}{C_i}.\frac{\mu_{i-1}}{\mu_i}\delta x$$

and
$$\rho_i^2\delta q_i=\frac{C_{i-1}}{C_i}.\mu_{i-1}.\mu_i R_i^2\delta x;$$

but it may be easily seen that

$$\mu_{i-1}.\mu_i=\frac{1}{C_{i-1}^2};\ \text{except when } i=1, \text{ for which case } \mu_{i-1}.\mu_i=1,$$

hence
$$\rho_i^2\delta q_i=\frac{1}{C_{i-1}.C_i}R_i^2\delta x, \text{ when } i>1, \text{ and } =\frac{C_0}{C_1}R_1^2\delta x \text{ when } i=1,$$

which proves the theorem in the text.

certain of the residues may be wanting, which will be the case when any of the secondary Bezoutics have their first or successive first terms affected with the coefficient zero; the equation connecting the residues with the Jacobian will then change its form (as some of the quantities $C_1, C_2, \ldots C_n$ will become zero); but I do not propose to enter for the present into the theory of these failing, or as they may more properly be termed, Singular cases in the theory of elimination.

Art. (*m*.) The series last obtained for J (f, g) leads to a result of much interest in the theory, and of which great use is made in the concluding section of this memoir, viz. the identification of the Jacobian (abstraction made of the numerical factor n) with what the Bezoutiant becomes when in place of the n variables in it, $u_1, u_2 \ldots u_n$, we write $x^{n-1}, x^{n-2}, \ldots x, 1$. Thus suppose f and g to be each of the third degree, and let

$$Ax^3+Hx+G$$
$$Hx^3+Bx+F$$
$$Gx^3+Fx+C$$

be the three primary Bezoutics; if we make

$$x^2=u \quad x=v \quad 1=w,$$

these may be written under the form

$$Au+Hv+Gw=L$$
$$Hu+Bv+Fw=M$$
$$Gu+Fv+Cw=N;$$

and if the Bezoutiant be called $\mathfrak{H}$, we have

$$L=\frac{d\mathfrak{H}}{du} \quad M=\frac{d\mathfrak{H}}{dv} \quad N=\frac{d\mathfrak{H}}{dw}.$$

The simplified residues to f and g are L, (L, M), (L, M, N), where (L, M) means the result of eliminating u between L and M, and (L, M, N) the result of eliminating u and v between L, M, N; and by a theorem (virtually implied in the direct method* of reducing a quadratic function to the form of a sum of squares), if we call the leading coefficients of these quantities C_1, C_2, C_3, we have

$$\frac{L^2}{C}+\frac{(L, M)^2}{C_1 . C_2}+\frac{(L, M, N)^2}{C_2 . C_3}=\mathfrak{H}.$$

Hence when $n=3\frac{1}{3}.J(f, g)=\mathfrak{H}$ when in $\mathfrak{H}$, u, v, w are turned into $x^2, x, 1$, and so in general for any values of n, the Bezoutiant correspondingly modified, becomes $\frac{1}{n}.J(f, g)$, as was to be shown†.

* Viz. that of M. CAUCHY, adverted to in Section IV. art. 44–45.

† Compare JACOBI, "De Eliminatione," § 2. The general expression for the allotrious factor, I may here incidentally mention, is given under the head Theorem *a*, § 16, which comes quite at the end of the same paper.

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Art. (*n*.) The expressions obtained for the quotients to $\frac{f'x}{fx}$ may be generalized and extended to the quotients to $\frac{\varphi x}{fx}$, where φx and fx are two functions of x of any degrees m and n, whose roots are respectively, $k_1\ k_2 \ldots k_m$, and $h_1\ h_2 \ldots h_n$. If we suppose

$$\frac{\varphi x}{fx}=\frac{1}{\mathrm{Q}(x)-}\,\frac{1}{q_2(x)-}\,\frac{1}{q_3(x)-}\,\cdots\cdots\,\frac{1}{q_{m+1}(x)},$$

where $\mathrm{Q}(x)$ is of $n-m$ dimensions, and $q_2(x)$, $q_3(x) \ldots q_{m+1}(x)$, each of one dimension in x, it may be proved that on writing

$$\frac{1}{\mathrm{Q}(x)-}\,\frac{1}{q_2(x)-}\,\cdots\cdots\,\frac{1}{q_i(x)}=\frac{\mathrm{N}_i(x)}{\mathrm{D}_i(x)},$$

we shall have

$$\Sigma_m^1\theta\left\{(\mathrm{N}_i\,k_\theta)^2\cdot\frac{fk_\theta}{\varphi' k_\theta}(x-k_\theta)\right\}=\mathrm{C}q_{i+1}(x) \quad \ldots\ldots\ldots \quad (\mathrm{A}.)$$

$$\Sigma_n^1\theta\left\{(\mathrm{D}_i\,h_\theta)^2\cdot\frac{\varphi h_\theta}{f' h_\theta}(x-h_\theta)\right\}=\mathrm{C}'q_{i+1}(x) \quad \ldots\ldots\ldots \quad (\mathrm{B}.)$$

where

$$\mathrm{C}\pm\mathrm{C}'=0, \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathrm{E}.)$$

$\mathrm{C}q_{i+1}(x)$ being the $(i+1)$th simplified quotient. When $\mathrm{Q}(x)$ is a linear function of x, in finding $q_1 x$ from the formula B, we must take $\mathrm{D}_0 x=1$. The proof of this theorem being generally true, may easily be shown to depend upon its being true in the special case*, when $m=\mu+i$, and $n=\mu+i'$ (m being supposed less than n), and $h_1, h_2, \ldots h_n$ become $l_1\ l_2 \ldots l_\mu\ h_1\ h_2 \ldots h_{i'}$, and $k_1\ k_2 \ldots k_m$ become $l_1\ l_2 \ldots l_\mu\ k_1\ k_2 \ldots k_i$; and the truth of the theorem for this special case (if for instance we wish to prove the formula (B)) depends upon the expression

$$\left|\begin{matrix} h_1 h_2 \ldots h_{i'-1} \\ k_1 k_2 \ldots k_m \end{matrix}\right| \div \left|\begin{matrix} h_1 h_2 & \ldots h_{i'-1} \\ h_{i'} h_{i'+1} \ldots h_n & \end{matrix}\right| \times \left|\begin{matrix} h_1 h_2 \ldots h_{i'} \\ k_1 k_2 \ldots k_m \end{matrix}\right| \div \left|\begin{matrix} h_1 h_2 & \ldots h_{i'} \\ h_{i'+1} h_{i'+2} \ldots h_n & \end{matrix}\right|$$

being identical with the expression

$$\left\{\left|\begin{matrix} h_1 h_2 \ldots h_{i'-1} \\ k_1 k_2 \ldots k_m \end{matrix}\right| \div \left|\begin{matrix} h_1 h_2 & \ldots h_{i'-1} \\ h_{i'} h_{i'+1} \ldots h_n & \end{matrix}\right| \times (h_{i'}-h_1)(h_{i'}-h_2)\ldots(h_{i'}-h_{i'-1})\right\} \times \frac{\left|\begin{matrix} h_{i'} \\ k_1 k_2 \ldots k_m \end{matrix}\right|}{\left|\begin{matrix} h_{i'} \\ h_1 h_2 \ldots h_{i'-1} h_{i'+1} \ldots h_n \end{matrix}\right|}$$

* By virtue of the Lemma, that when φx and fx are two algebraical functions (x^m+ax^n &c.); ($x^{m+e}+ax^{m+(e)}$ &c.) no function of the coefficients vanishing identically when i roots of fx coincide with i roots of φx respectively can be formed, in which there are fewer of the coefficients of f and φ respectively than appear in the leading coefficient of the $(n-i+1)$th residue of $\frac{\varphi}{f}$.

as it may readily be shown to be. And the formula (A.) may be verified in precisely the same manner. There is no difficulty in finding the values of C and C', which are products of powers, some positive and some negative, of the leading coefficients in the simplified residues, and recognising that they satisfy the equation (E.); when φx is of one degree below fx this equation is of the form $C+C'=0$.

Art. (*o.*) When $\varphi x=f'x$, this expression for the $(i+1)$th simplified quotient becomes $\Sigma(D_i h)^s(x-h)$, as previously found; the correlative expression will be

$$-\Sigma(N_i k)^s \frac{fk}{f''k}(x-k),$$

k being any root of $f'x=0$, which is equal to the former expression. The general expressions above given for the simplified quantities are of course integral functions of h and k, although given under the form of the sums of fractions, by virtue of the well-known theorem that $\Sigma\frac{\mathfrak{I}(h)}{f'h}$, where $\mathfrak{I}$ is an integral function of h, and the summation comprises all the roots (h) of $fh=0$ is always integral.

Art. (*p.*) It will be found that for all values of i greater than unity

$$\Sigma'_m\theta(N_i.k_\theta)\frac{fk_\theta}{\varphi'k_\theta}=0,$$

and that

$$\Sigma'_n\theta.(D_i h_\theta)\frac{\varphi h_\theta}{f'h_\theta}=0.$$

The theorem of art (*n.*) is in effect a theorem of Cumulants of the form

$$[Q_1(x), q_2(x), \ldots q_i(x)\ldots q_n(x)],$$

where the elements are all independent of one another, and where fx represents

$$[Q_1(x)\, q_2(x)\, q_3(x)\ldots q_n(x)]$$ and φx represents $[q_x x, q_3(x), \ldots q_n(x)]$,

n being any *number whatever greater* than i; this makes the theorem still more remarkable. The urgency of the press precludes my investigating for the present the more general theorem which must be presumed to exist, whereby q_{i+1} can be connected with $[q_1 q_2 q_3\ldots q_i]$, or $[q_2 q_3\ldots q_i]$, and with $[q_1 q_2 q_3\ldots q_{i+e}]$ and $[q_2 q_3\ldots q_{i+e}]$, when each (q) represents a function of an *arbitrary* degree in x. The theorem so generalized would comprehend the complete theory of the quotients arising from the process of continued division, without exclusion of the singular cases (at present supposed to be excluded) where one or several consecutive principal coefficients in one or more of the residues, vanish.

Art. (*q.*) The complete statement of two twin theorems suggested by and intimately connected with the biform representation of the quotients $\frac{\varphi x}{fx}$, given in the preceding article, is too remarkable to be omitted.

Suppose $\varphi x=f'x$, and let the successive convergents to $\frac{f'x}{fx}$ be called

$$\frac{1}{T_1.x}, \frac{t_1 x}{T_2 x}, \ldots \frac{t_{n-2}.x}{T_{n-1}.x}, \frac{t_{n-1}.x}{T_n x}$$

where the subscrolet index to t or T indicates the degree in x. Then if we call the

roots of fx h_1 $h_2 \ldots h_n$ the theorem already cited in a preceding article, concerning the denominators of the convergents, may be expressed as follows:—

$$\left\{\begin{array}{llll} \left(\frac{f'h_1}{\varphi h_1}\right)^2; & \left(\frac{f'h_2}{\varphi h_2}\right)^2; & \ldots; & \left(\frac{f'h_n}{\varphi h_n}\right)^2 \\ (\mathrm{T}_1\, h_1)^2; & (\mathrm{T}_1\, h_2)^2; & \ldots; & (\mathrm{T}_1\, h_n)^2 \\ (\mathrm{T}_2\, h_1)^2; & (\mathrm{T}_2\, h_2)^2; & \ldots; & (\mathrm{T}_2\, h_n)^2 \\ \quad \ldots & \ldots & \ldots & \ldots \\ (\mathrm{T}_{n-1}.h_1)^2; & (\mathrm{T}_{n-1}.h_2)^2; & \ldots; & (\mathrm{T}_{n-1}.h_n)^2 \end{array}\right\} = 0$$

where it will be observed that the first line of terms consists exclusively of units, since $f'x = \varphi x$ by hypothesis.

Correlatively I have ascertained that preserving the same assumption that $\varphi x = f'x$, so that consequently $\frac{\varphi' k}{fk}$ means $\frac{f'' k}{fk}$, the following theorem obtains, viz. that if k_1 $k_2 \ldots k_{n-1}$ are the $(x-1)$ roots of φx.

$$\left\{\begin{array}{llll} \left(\frac{\varphi' k_1}{fk_1}\right)^2; & \left(\frac{\varphi' k_2}{fk_2}\right)^2; & \ldots & \left(\frac{\varphi' k_{n-1}}{f k_{n-1}}\right)^2 \\ (t_1(k_1))^2; & (t_1(k^2))^2; & \ldots & (t_1(k_{n-1}))^2 \\ (t_2(k_1))^2; & (t_2(k_2))^2; & \ldots & (t_2(k_{n-1}))^2 \\ \quad \ldots & \ldots & \ldots & \ldots \\ (t_{n-2}(k_1))^2; & (t_{n-2}(k_2))^2; & \ldots & (t_{n-2}(k_{n-1}))^2 \end{array}\right\} = 0.$$

It may consequently be conjectured, when φ and f are independent functions of x and respectively of the degree $n-1$ and n, and $\frac{\varphi x}{fx}$ is expanded under the form of a continued fraction, of which, as before, $\frac{1}{\mathrm{T}_1}$; $\frac{t_1}{\mathrm{T}_2}$; ...; $\frac{t_{n-1}}{\mathrm{T}_n}$ are the successive convergents, that we shall have analogous determinants to the twin forms above given, each separately vanishing, these more general determinants differing only from their model forms in respect of the uppermost line of terms in the one of them, being each multiplied by certain functions of h_1, $h_2 \ldots h_n$ respectively (all of which become units when $\varphi x = f'x$), and in the other of them by certain functions of k_1, $k_2, \ldots k_n$.

The exact form, however, of such functions, and even the possibility of such form being found capable of making the determinants vanish, remains open for further inquiry.

Section IV.

On some further Formulæ connected with M. Sturm's *theorem, and on the Theory of Intercalations whereof that theorem may be treated as a corollary.*

As preparatory to some remarks about to be made on the formulæ connected with M. Sturm's theorem, it is necessary to premise two theorems concerning quadratic functions of great importance, one which, notwithstanding its extreme simplicity, is as far as I know very little (if at all) known, and the other was given in part many years ago by M. Cauchy, but is also not generally known. The former of these

two theorems is as follows. If a quadratic homogeneous function of any number of variables be (as it may be in an infinite variety of ways) transformed into a function of a new set of variables, linearly connected by real coefficients with the original set, in such a way that only positive and negative squares of the new variables appear in the transformed expression, the number of such positive and negative squares respectively will be constant for a given function whatever be the linear transformations employed. This evidently amounts to the proposition, that if we have $2n$ positive and negative squares of homogeneous real linear functions of n variables identically equal to zero, the number of positive squares and of negative squares must be equal to one another, so that *ex. gr.* we cannot have

$$\pm\{u_1^2+u_2^2+\&c.\ldots+u_n^2+u_{n+1}^2-u_{n+2}^2-u_{n+3}^2-\&c.\ -u_{2n}^2\}$$

identically zero when n of the variables are linear functions of the remaining n; and this is obviously the case, for if the equation could be identically satisfied we might make

$$u_{n+2}=u_1 \quad u_{n+3}=u_2\ldots\ldots u_{2n}=u_{n-1},$$

and we should then be able to find u_{n+1} as a real numerical multiple of u_n, and consequently should have the equation $u_n^2.\{1+k^2\}=0$, which is obviously impossible; *à fortiori* we may prove that in the identical equation existing between the sum of an even number of positive and of negative squares of real linear functions of half the number of independent variables, there cannot be *more* than a difference of two (as we have proved that there cannot be that difference) between the number of positive and negative squares. Hence there must be as many of one as of the other; and as a consequence, the number of positive squares or of negative squares in the transform of a given quadratic function of any number of variables effected by any set of real linear substitutions is constant, being in fact some unknown transcendental function of the coefficients of the given function. I quote this law (which I have enunciated before, but of which I for the first time publish the proof) under the name of the law of inertia for quadratic forms.

Art. (45.). The other theorem is the following. If any quadratic function be represented in the umbral notation* under the form of $(a_1x_1+a_2x_2+\ldots+a_nx_n)^2$, where a_1, $a_2\ldots a_n$ are the umbræ of the coefficients, and x_1, $x_2\ldots x_n$ the variables, then by writing

$$\begin{vmatrix}a_1\\a_1\end{vmatrix}x_1+\begin{vmatrix}a_1\\a_2\end{vmatrix}x_2+\begin{vmatrix}a_1\\a_3\end{vmatrix}x_3+\begin{vmatrix}a_1\\a_4\end{vmatrix}x_4+\ldots+\begin{vmatrix}a_1\\a_n\end{vmatrix}x_n=y_1$$

$$\begin{vmatrix}a_1&a_2\\a_1&a_2\end{vmatrix}x_2+\begin{vmatrix}a_1&a_2\\a_1&a_3\end{vmatrix}x+\begin{vmatrix}a_1&a_2\\a_1&a_4\end{vmatrix}x_4+\&c.+\begin{vmatrix}a_1&a_2\\a_1&a_n\end{vmatrix}x_n=y_2$$

$$\begin{vmatrix}a_1&a_2&a_3\\a_1&a_2&a_3\end{vmatrix}x_3-\begin{vmatrix}a_1&a_2&a_3\\a_1&a_2&a_4\end{vmatrix}x_4+\&c.+\begin{vmatrix}a_1&a_2&a_3\\a_1&a_2&a_n\end{vmatrix}x_n=y_3$$

$$\&c.\ \&c.\ \&c.$$

$$\begin{vmatrix}a_1&a_2\ldots a_n\\a_1&a_2\ldots a_n\end{vmatrix}x_n=y_n,$$

* For an explanation of the umbral notation, see London and Edinburgh Philosophical Magazine, April 1851, or thereabouts.

$(a_1x_1+a_2x_2+\ldots+a_nx_n)^2$ will assume the form

$$\begin{vmatrix} a_1 \\ a_1 \end{vmatrix} y_1^2 + \frac{\begin{vmatrix} a_1 a_2 \\ a_1 a_2 \end{vmatrix}}{\begin{vmatrix} a_1 \\ a_1 \end{vmatrix}} y_2^2 + \frac{\begin{vmatrix} a_1 a_2 a_3 \\ a_1 a_2 a_3 \end{vmatrix}}{\begin{vmatrix} a_1 a_2 \\ a_1 a_2 \end{vmatrix}} y_3^2 + \ldots + \frac{\begin{vmatrix} a_1 a_2 \ldots a_{n-1} a_n \\ a_1 a_2 \ldots a_{n-1} a_n \end{vmatrix}}{\begin{vmatrix} a_1 a_2 \ldots a_{n-1} \\ a_1 a_2 \ldots a_{n-1} \end{vmatrix}} y_n^2$$

and consequently the number of positive squares in the reduced form of the given function will always be the number of continuations or permanencies of sign of the series

$$1 \; ; \; \begin{matrix} a_1 \\ a_1 \end{matrix} \; ; \; \begin{vmatrix} a_1 a_2 \\ a_1 a_2 \end{vmatrix} \; ; \; \begin{vmatrix} a_1 a_2 a_3 \\ a_1 a_2 a_3 \end{vmatrix} \; ; \; \ldots \begin{vmatrix} a_1 a_2 \ldots a_n \\ a_1 a_2 \ldots a_n \end{vmatrix}$$

the several terms of this progression being in fact the determinants of what the given function becomes when we obliterate successively all the variables but one, then all but that another, then all but these two and a third, until finally, the last term is the determinant of the given function with all the variables retained. This comes to saying that if we call the function (suppose of four variables) f, and we write

$$\begin{matrix} \frac{d^2f}{dx_1^2} & \frac{d^2f}{dx_1 dx_2} & \frac{d^2f}{dx_1 dx_3} & \frac{d^2f}{dx_1 dx_4} \\ \frac{d^2f}{dx_2 dx_1} & \frac{d^2f}{dx_2^2} & \frac{d^2f}{dx_2 dx_3} & \frac{d^2f}{dx_2 dx_4} \\ \frac{d^2f}{dx_3 dx_1} & \frac{d^2f}{dx_3 dx_2} & \frac{d^2f}{dx_3^2} & \frac{d^2f}{dx_3 dx_4} \\ \frac{d^2f}{dx_4 dx_1} & \frac{d^2f}{dx_4 dx_2} & \frac{d^2f}{dx_4 dx_3} & \frac{d^2f}{dx_4^2}, \end{matrix}$$

(where all the terms are of course coefficients of the given function expressed as above for greater symmetry of notation), the inertia of f will be measured by the number of continuations of sign in the series formed of the *successive principal minor coaxal* determinants $\left(\text{in writing which I shall use in general } (r, s) \text{ to denote } \frac{d^2f}{dx_r dx_s}\right)$,

$$1, \; (1,1), \; \begin{bmatrix} (1,1) & (1,2) \\ (2,1) & (2,2) \end{bmatrix}, \; \begin{bmatrix} 1,1 & 1,2 & 1,3 \\ 2,1 & 2,2 & 2,3 \\ 3,1 & 3,2 & 3,3 \end{bmatrix}, \; \begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{bmatrix}$$

and in like manner in general*.

* I have given a direct *à posteriori* demonstration in the London and Edinburgh Philosophical Magazine, that the number of continuations of sign in any series formed like the above form a symmetrical matrix, is unaffected by any permutations of the lines and columns thereof, which leaves the symmetry subsisting, that is to say (using the umbral notation), if $\theta_1, \theta_2, \theta_3 \ldots \theta_i$ are disjunctively equal, each to each, in any arbitrary order to 1, 2, 3....i, the number of continuations of sign in the series

$$1, \begin{vmatrix} a_{\theta_1} \\ a_{\theta_1} \end{vmatrix}, \begin{vmatrix} a_{\theta_1} a_{\theta_2} \\ a_{\theta_1} a_{\theta_2} \end{vmatrix}, \begin{vmatrix} a_{\theta_1} a_{\theta_2} a_{\theta_3} \\ a_{\theta_1} a_{\theta_2} a_{\theta_3} \end{vmatrix}, \ldots, \begin{vmatrix} a_{\theta_1} a_{\theta_2} a_{\theta_3} \ldots a_{\theta_i} \\ a_{\theta_1} a_{\theta_2} a_{\theta_3} \ldots a_{\theta_i} \end{vmatrix}$$

is irrespective of the order of the natural numbers 1, 2, 3....i in the arrangement $\theta_1, \theta_2, \theta_3 \ldots \theta_i$.

Art. (46.). Reverting now to the simplified Sturmian residues, since by the theory set out in the first Section these differ from the unsimplified complete residues required by the Sturmian method only in the circumstance of their being divested of factors, which are necessarily perfect squares and therefore essentially positive, these simplified Sturmians may of course be substituted for the complete Sturmians for the purposes of M. STURM's theorem. The leading coefficients in these simplified Sturmians, reckoning $f'(x)$ as one of them, will be

$$m\Sigma\zeta(h_1 h_2),\ \Sigma\zeta(h_1 h_2 h_3)\ldots\zeta(h_1 h_2\ldots h_m),$$

which it is easily seen, as remarked long ago by Mr. CAYLEY, are the successive principal minor coaxal determinants of the matrix

$$\begin{matrix}\sigma_0, \sigma_1, \sigma_2, \sigma_3\ldots\sigma_{m-1}\\ \sigma_1, \sigma_2, \sigma_3\ldots\ldots\sigma_m\\ \sigma_2, \sigma_3\ldots\ldots\ldots\sigma_{m-1}\\ \ldots\ldots\\ \sigma_{m-1}\sigma_m\ldots\ldots\sigma_{2m-2},\end{matrix}$$

where in general $\sigma_r = h_1^r + h_2^r + \ldots + h_m^r$, and of course $\sigma_0 = m$. M. HERMITE has improved upon this remark by observing, which is immediately obvious, that if we use σ_r to denote, not the quantity above written, but $\frac{h_1^r}{x-h_1} + \frac{h_2^r}{x-h_2} + \ldots \frac{h_m^r}{x-h_m}$, the successive coaxal determinants of the above matrix will become respectively

$$\Sigma\frac{1}{x-h_1};\ \Sigma\left\{\frac{\zeta(h_1 h_2)}{(x-h_1)(x-h_2)}\right\};\ \Sigma\frac{\zeta(h_1 h_2 h_3)}{(x-h_1)(x-h_2)(x-h_3)};\ \ldots\frac{\zeta(h_1 h_2\ldots h_m)}{(x-h_1)(x-h_2)\ldots(x-h_m)};$$

that is to say, these successive coaxal determinants, when multiplied up by fx, will become respectively

$$\Sigma(x-h_2)(x-h_3)\ldots(x-h_m);\ \Sigma\zeta(h_1 h_2)\{(x-h_3)(x-h_4)\ldots(x-h_m)\};\ \ldots\ldots;\ \Sigma\zeta(h_1 h_2\ldots h_m),$$

that is to say, will represent the simplified Sturmian series given by my general formulæ. M. HERMITE further remarks, that the matrix formed after this rule will evidently be that which represents the determinant of the quadratic function (which may be treated as a generating function)

$$\Sigma\frac{1}{x-h_1}.\{u_1 + h_1.u_2 + h_1^2.u_3 + \ldots + h_1^{m-1}.u_n\}^2,$$

in which, since only the squared differences of the terms in the (h) series finally remain in the successive coaxal determinants, we may write $(x-h_1)$, $(x-h_2)\ldots(x-h_m)$ simultaneously in place of $h_1 h_2\ldots h_m$ without affecting the result, consequently the generating function above may be replaced by the generating function

$$\Sigma\frac{1}{x-h_1}.\{u_1 + (x-h_1)u_2 + (x-h_1)^2.u_3 + \ldots + (x-h_1)^{m-1}.u_m\}^2;$$

the corresponding matrix to which becomes

$$\Sigma\frac{1}{x-h_1},\quad \begin{matrix} \theta_0, & \theta_1, & \theta_2, \ldots \theta_{m-2} \\ \theta_1, & \theta_2, & \theta_3, \ldots \theta_{m-1} \\ \theta_2, & \theta_3, & \ldots \theta_m \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \theta_{m-2}, & \theta_{m-1}, & \ldots \theta_{2m-3}, \end{matrix}$$

where θ_i denotes $\Sigma(x-a)^i$, and $\Sigma\frac{1}{x-h_1}=\frac{f'x}{fx}$. Hence every simplified residue is of the form

$$f'x\times\begin{Vmatrix} \theta_1 & \theta_2 & \ldots\theta_r \\ \theta_2 & \theta_3 & \ldots\theta_{r+1} \\ \cdot & \cdot & \cdot \\ \theta_r & \theta_{r+1} & \ldots\theta_{2r-1} \end{Vmatrix}+fx\times\begin{Vmatrix} 0 & \theta_0 & \theta_1\ldots\theta_r \\ \theta_0 & \theta_1 & \ldots\theta_{r+1} \\ \cdot & \cdot & \cdot \\ \theta_r & \theta_{r+1} & \ldots\theta_{2r-1} \end{Vmatrix}.$$

The residue in question will be of the degree $m-r-2$ in x, and consequently we have, according to the notation antecedently used for the syzygetic equations

$$t_{r+1}=\begin{Vmatrix} \theta_1 & \theta_2 & \ldots\theta_r \\ \theta_2 & \theta_3 & \ldots\theta_{r+1} \\ \cdot & \cdot & \cdot \\ \theta_r & \theta_{r+1} & \ldots\theta_{2r-1} \end{Vmatrix}$$

$$-\tau_r=\begin{Vmatrix} 0 & \theta_0 & \theta_1\ldots\theta_r \\ \theta_0 & \theta_1 & \ldots\theta_{r+1} \\ \theta_1 & & \ldots\theta_{r+2} \\ \cdot & \cdot & \cdot \\ \theta_r & \theta_{r+1} & \ldots\theta_{2r-1} \end{Vmatrix}.$$

Elegant and valuable for certain purposes as are these formulæ for t_{r+1} and τ_r, they are affected with the disadvantage of being expressed by means of formulæ of a much higher degree in the variable x than really appertains to them, the paradox (if it may be termed such) being explained by the circumstance of the coefficients of all the powers of x above the right degree being made up of terms which mutually destroy one another. Upon the face of the formulæ, t_{r+1} and τ_r which are in fact only of the degrees $r+1$, and r respectively in x would appear to be of the degree $1+3+5+\ldots+(2r-1)$, *i. e.* of the degree r^2.

Art. (47.). I may add the important remark, which does not appear to have occurred immediately to my friend M. Hermite when he communicated to me the above most interesting results, that in fact, by virtue of the law of inertia for quadratic forms, we may dispense with any identification of the successive coaxal determinants of the matrix to the generating function

$$\Sigma\frac{1}{\xi-h_1}(u_1+h_1u_2+h_1^2.u_3+\ldots+h_1^{m-1}.u_m)^2$$

with my formulæ for the Sturmian functions, and prove *ab initio* in the most simple manner, that the successive ascending coaxal determinants (always of course supposed to be taken about the axis of symmetry) of the matrix to the form above written, or to the more general form (which I shall quote as G, viz.)

$$\Sigma(\varrho-h_1)^q\{\varphi_1.(h_1)u_1+\varphi_2(h_1)u_2+\ldots+\varphi_m.(h_m).u_m\}^2 \quad . \quad . \quad . \quad . \quad . \quad \text{(G.)}$$

(where φ_1, $\varphi_2 \ldots \varphi_m$ are absolutely arbitrary integral forms of function with real coefficients), will form a rhizoristic series in regard to fx (*i. e.* a series, the difference between the number of the continuations of sign between the successive terms of which corresponding to two different values of ϱ will determine the number of real roots of ϱ lying between such two assumed values), provided only that q be an odd positive or negative integer. Nothing can be easier than the demonstration, for whenever ϱ is greater than any one of the real roots as (h_1)

1st. Any pair of imaginary roots will give rise to two terms of the form

$$(l+m\sqrt{-1})^q.(v+w\sqrt{-1})^2 \text{ and } (l-m\sqrt{-1})^q.(v-w\sqrt{-1})^2;$$

or more simply,

$$(\mathrm{L}+\mathrm{M}\sqrt{-1}).(v^2+w^2+2vw\sqrt{-1})$$

and

$$(\mathrm{L}-\mathrm{M}\sqrt{-1}).(v^2-w^2-2vw\sqrt{-1}),$$

where v and w are real linear functions of $u_1, u_2, \ldots u_n$. The sum of which couple will be

$$2\{\mathrm{L}.(u^2-v^2)-2\mathrm{M}uv\}=\frac{2}{\mathrm{L}}.\{(\mathrm{L}u-\mathrm{M}v)^2-(\mathrm{L}^2+\mathrm{M}^2)v^2\}=p^2-q^2;$$

so that each such couple combined will for every value of x give rise to one positive and one negative square.

2ndly. Any real root of the series $h_1, h_2, \ldots h_m$, when ϱ is taken greater than such root, will give rise to a positive square of a real linear function of $u_1, u_2, \ldots u_n$.

3rdly. Any real root of the same series, when ϱ is beneath it in value (q being odd), will give rise to the negative of the square of a real linear function of the same. Hence the number of real roots between ϱ taken equal to one value (a), and ϱ taken equal to any other value (b), will be denoted by the loss of an equal number of positive squares in the reduced form of the expression (G.) when ϱ is taken (a) and when ϱ is taken (b); *i. e.* by virtue of art. (45.) will be denoted by the difference of the number of permanencies of sign in the successive minor determinants of the matrix corresponding to the quadratic form (G.)* (which we have taken as our generating function) resulting

* The *inertia* of the quadratic form G is the measure of the number of real roots of fx comprised between ∞ and ρ, and may be estimated in any manner that may be found most convenient. If ρ be made infinity, and $\varphi_i h$ be taken equal to h^{i-1}, and the inertia of the corresponding value of G be estimated by means of the formulæ in ordinary use by geometers for determining the nature of a surface of the second degree, the criteria of the number of real roots in fx will be, or may be made to be, symmetrical in respect to the two ends of the expression fx. This system of criteria, however, is not so good as that given by the Bezoutiant to the two differential coefficients of $f(x, 1)$ taken with regard to x and 1 respectively, which will also possess the like character of symmetrical indifference, and be one less in number than the former.

from the substitution respectively of a and b in place of ϱ, which gives a theorem equivalent to that of M. STURM, transformed by my formulæ, when we choose to adopt the particular suppositions

$$q=-1 \quad \varphi_1 h=1 \quad \varphi_2 h=h \quad \varphi_3 . h=h^2 \ldots \varphi_m h=h^{m-1}.$$

This method of *constructing* a rhizoristic series to fx by a direct process is deserving of particular attention, because it does not involve the use of the notion of continuous variation, upon which all preceding proofs of STURM's theorem proceed. It completes the cycle of the Sturmian ideas. Happily this cycle was commenced from the other end, for it would have been difficult to have suspected that the root-expressions for the terms in the rhizoristic series could be identified with the residues, had the former been the first to be discovered, and much of the theory of algebraical common measure laid open by means of this identification would probably have remained unknown.

Art. (48.). I proceed now to consider a theorem concerning the relative positions of the real roots of two independent algebraical functions as indicated by the succession of signs presented by their Bezoutian secondaries; this more general theory of intercalations or relative interpositions will be seen to include within it as a corollary the justly celebrated theorem of M. STURM.

Let the real roots of fx taken in descending order of magnitudes be $h_1 h_2 \ldots h_p$, and the real roots of φx taken in the like order $\eta_1 \eta_2 \ldots \eta_q$, so that

$$fx=(x-h_1)(x-h_2)\ldots(x-h_p)\mathrm{H}$$
$$\varphi x=(x-\eta_1)(x-\eta_2)\ldots(x-\eta_q)\mathrm{K},$$

H and K being functions of x incapable of changing their signs. Now, as in M. STURM's method, let us inquire what takes place in respect to the sign of $\frac{\varphi(x)}{f(x)}$, which I shall call the Indicatrix, as x descends the scale of real magnitude from $+\infty$ to $-\infty$. If between $+\infty$ and h_1, i real roots of φx are contained, it is obvious that as x travels from $+\infty$ to the superior brink of h_1, the Indicatrix will change its sign from $+$ to $-$ and from $-$ to $+$ altogether i times, so that at the moment when x is about to pass through h_1, it will be positive if i is zero or even, and negative if i is odd; but the moment after x has passed through the value h_1, the indicatrix will be negative on the first supposition, and positive on the other supposition. Hence immediately after the passage of x through h_1 the indicatrix will have been once oftener negative than positive on the one supposition, and as often negative as positive on the other. Again, in like manner as x traverses the interval between h_1 and the inferior brink of h_2, if no η or an even number of η's occupy this interval, the sign which the Indicatrix had at the beginning of this interval will have been reversed once oftener than restored; but if there be an odd number of k's so interposed, the number of reversals and restorations will have been identical; and so for each successive interval, reckoned from a value for x immediately subsequent to one real root of fx, down to a value immediately subsequent to the next less real root of the

same; and it is evident that the effect upon the sign of the Indicatrix at the end of every such interval depends, not upon the number of η's grouped together in such interval, but upon the form of the group as regards its being made up of an odd or even number of terms [the first interval will of course be understood to extend from $+\infty$ to a value immediately inferior to h_1, and the last from a value immediately inferior to h_r to $-\infty$]. Hence as regards the relation of the signs of the Indicatrix at the beginning to the sign at the end of every such interval, nothing will be altered by taking away any even number of η's that may be found therein. If we suppose this to be done, we shall then have in some of the intervals one η occurring and in the other intervals no η; that is to say, some of the h's will be separated by single η's, but other h's will come together. Again, by removing any even number of h's not separated by η's (and thus removing an even number of intervals), it is clear that as many changes of sign of the Indicatrix will have been done away with from $+$ to $-$ as from $-$ to $+$, and no effect upon the excess of the one kind of changes of sign over the other kind of changes of sign will have been produced. By removing pairs of h's in this manner, it may happen that η's will again be brought together, any even number of which, not separated by h's, may again be removed and then pairs of h's not separated by η's in their turn, and so continually *toties quoties* until at length we must arrive at a reduced system of h's and η's, where no two h's and no two η's come together, or else all the h's and all the η's will have disappeared. Let the scale of h's and η's thus simplified and reduced be called the effective scale of intercalations. The number of h's and the number of η's in any such scale will be equal, or will at most differ from one another by a unit, since at each part of the scale, except at the end, every h is followed by an η and every η by an h. If the scale begins and ends with an h, there will of course be one more h than η; if it begin and end with an η, there will be one more η than h; if it begin with an h or an η and end with an η or h, there will be as many of the one as of the other.

1st. Suppose the effective intercalation scale to commence with an h; then in passing from $+\infty$ to just beyond the first h the sign of the indicatrix $\frac{\varphi x}{fx}$ changes from $+$ to $-$; it changes again from $-$ to $+$ as it passes the first η, then again from $+$ to $-$ as it passes the second h, and so on; that is to say, there will be a change always in the same direction from $+$ to $-$ as x passes, from being just greater than to being just less than any h appearing in the effective scale. 2nd. If the effective scale begin with η, the indicatrix will conversely be negative after passing the first and every subsequent η, and change from $-$ to $+$ in the act of passing through the first and every subsequent h. So that on either supposition the changes of sign for the effective scale always take place in the same direction, and the number of h's in the effective scale will be measured by the number of such changes, and consequently will be measured by the difference between the number of times that the indicatrix $\frac{\varphi x}{fx}$ changes its sign from $+$ to $-$ as x passes through each in turn of the real roots of fx, and the number of times that in passing through any such root it changes its sign from $-$ to $+$; if the former number be

greater than the latter, the effective scale of interpositions will begin with a root of fx; if it be less, the scale will begin with a root of φx. If instead of beginning with $+\infty$ and ending with $-\infty$ we begin and end with any two limits, a and b respectively (making abstraction of all roots of fx or of φx lying outside these limits, and forming the effective intercalation scale with the roots comprised within these limits exclusively), we shall obviously obtain a similar result, but with the condition that the changes from $+$ to $-$ will be *in excess* if an even number of h's and η's combined be cut off by the superior limit, and the effective scale begin with an h, or if an odd number of h's and η's combined be so cut off and the scale begin with an η; and in *defect* if an odd number of h's and η's combined be so cut off and the scale begin with an h, or an even number be so cut off and the scale begin with an η. If, now, supposing fx to be of n, and φx of not more than n, say (m) dimensions, we form the *signaletic* series fx, φx, B_1, B_2, ...B_m (where the B_1, B_2, ...B_m are the Bezoutian secondaries or simplified successive residues corresponding to $\frac{\varphi x}{fx}$ expanded under the form of an improper continued fraction), it may be shown, in the same way as for STURM's theorem, that whenever $\frac{\varphi x}{fx}$ changes from $+$ to $-$ a change of sign will be gained in the series, and when from $-$ to $+$ a change will be lost; and that no change can be gained or lost except as x passes through the successive real roots of fx. Hence the difference between the number of changes of sign in the above signaletic series when x is taken (a), and the number of the same when x is taken (b), will indicate the number of roots of fx remaining in the effective scale of interpositions formed between such of the roots of fx and of φx as lie between (a) and (b); calling the one number $\mathrm{I}(a)$ and the other $\mathrm{I}(b)$, the sign of $\mathrm{I}(b)-\mathrm{I}(a)$ depends not on the relative magnitudes of (a) and (b), but upon the manner in which the effective scale commences; if $\mathrm{I}(a)-\mathrm{I}(b)$ is positive, the effective scale formed between the (a) and (b) will commence with a root of fx; if negative, it will commence with a root of $\varphi(x)$.

Art. (49.). In forming the scale of effective interpositions, it is evidently not necessary to go on reducing the (h) series and the η series separately and alternately; the same result will be effected more expeditiously by eliding simultaneously any even number of h's that come together without being separated by an η, and any even number of η's that come together without being separated by an (h), and, repeating this process of simultaneous elision, as often as may be required, until no two h's or η's come together. Thus, for instance, denoting the magnitudes of the series of real roots of f and of φ by the distances of h and η points taken along a right line from a fixed point therein, and supposing such series of roots between the limits a and b to be

$$h\,h\,h\,\eta\,\eta\,\eta\,h\,\eta\,\eta\,h\,\eta\,\eta\,\eta\,h\,h\,\eta\,h\,\eta\,h\,h\,h\,h\,h\,\eta\,\eta\,h,$$

our first reduction brings this scale to the form

$$h\,\eta\,h\,h\,\eta\,\eta\,h\,\eta\,h\,h\,;$$

the next reduction brings it to the form

$$h\,\eta\,\eta\,\eta\,h\,\eta\,;$$

and a third and final reduction brings it to the form

$$h\,\eta\,h\,\eta;$$

and accordingly we shall find for such an arrangement of the h and η system

$$\mathrm{I}(b)-\mathrm{I}(a)=\pm 2.$$

Art. (50.). If we suppose $\varphi x=\frac{dfx}{dx}$, by a well-known theorem of algebra, any two consecutive roots of fx will contain between them an odd number of roots of φx, and the number of real roots of $f'x$ greater than the greatest root of fx, and the number of real roots of $f'x$ less than the least root of fx will each be even. Hence the effective intercalation scale between any two limits (a) and (b) will be formed by merely reducing the η groups to single units, and the number of h's in the scale so formed will be the total number of h's between the limits (a) and (b). Moreover, since such scale commences always with a root of fx, or with an even number of roots of $f'x$ followed by a root of fx, if the number of h's and η's cut off be even, and with a root of $f'x$ or an even number of roots of fx followed by a root of fx, if the number so cut off be odd, it follows that for this case $\mathrm{I}(a)-\mathrm{I}(b)$, (a) being the superior limit, will be always positive, and will measure the total number of real roots of $f(x)$ lying between (a) and (b); this, then, is STURM's theorem, treated as a corollary to the Theory of Intercalations.

Art. (51.). If we write down the last syzygetic equation between fx of m and $\varphi(x)$ of n dimensions, viz.

$$\tau_{n-1}.(x)f(x)-t_{m-1}(x)\varphi x+\vartheta_s=0,$$

it has been shown that the succession of signs in the series formed with fx, φx and their successive Bezoutian secondaries will contain the same number of continuations and variations as the series formed with $f(x)$, $t_{m-1}(x)$, and their successive Bezoutian secondaries. This indicates that the effective scale of interpositions for fx and φx will contain an equal number of roots of fx with the effective scale for fx and $t_{m-1}(x)$; the two scales however will not necessarily be identical, because the roots of φx will not necessarily be in the same order relative to the h's in the one scale as those of $t_{m-1}.x$ relative to the h's in the other scale. This equality is perfectly well explained *à posteriori* by the form of $t_{m-1}.x$, which by the formula in Section II. will be represented by

$$\Sigma(x-h_{q_1})(x-h_{q_2})..(x-h_{q_{m-1}}).\frac{\varphi h_{q_1}.\varphi h_{q_2}...\varphi h_{q_{m-1}}}{(h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})...h_{q_m}-h_{q_{m-1}}}.$$

Now, whenever x is indefinitely near to any one of the roots of fx, as h_{q_m}, this sum reduces to the simple expression

$$\varphi h_{q_1}\varphi h_{q_2}...\varphi h_{q_{m-1}}=\{\varphi h_1.\varphi h_2...\varphi h_m\}\frac{1}{\varphi h_{q_m}},$$

and consequently in the immediate neighbourhood of every real root of fx, $\varphi(x)$ and $t_{m-1}.x$ will have always the same or always a contrary sign, according as $\varphi h_{q_1}.\varphi h_{q_2}...\varphi h_{q_m}$ is positive or negative, which will depend upon the relative disposition of the real roots in f and φ; in either case the effective scale of interpositions for fx with φx and

for fx with $t_{n-1}.x$ must contain the same number of h's; but the difference will be, that if $\varphi h_1.\varphi h_2 \ldots \varphi h_m$ is positive an h will occupy the first place in each scale, or the second place in each scale; but if negative, then in one scale an (h) will occupy the first place, and in the other scale the second place.

Art. (52.). The same process of common measure or residues which serves to furnish a rhizoristic series for $f(x)$ or a synrhizoristic series for fx and φx, will serve also to furnish superior and inferior limits to the real roots of any proposed equation. Thus suppose fx to be any rational integral function of (x) of the degree (n) and $\varphi(x)$ any other function of x, which I shall begin with supposing to be of the degree ($n-1$), and let the successive quotients resulting from the process of finding the greatest common measure of fx, φx continued until the last remainder is not a constant but zero, be supposed to be (as they may generally be taken, but subject to cases of exception, which will hereafter be alluded to) n linear functions $q_1 q_2 \ldots q_n$, then we shall have

$$\frac{\varphi x}{fx} = \frac{1}{q_1+}\ \frac{1}{q_2+}\cdots\frac{1}{q_{n-1}+}\ \frac{1}{q_n},$$

and therefore

$$\varphi x = \mathrm{K.N}$$
$$fx = \mathrm{K.D},$$

where N is the numerator and D the denominator of the fraction

$$\frac{1}{q_1+}\ \frac{1}{q_2+}\cdots\frac{1}{q_n},$$

and K is a constant (the value of which is immaterial to be considered, but in fact equals

$$\pm\frac{L_0}{L_1^2}.\frac{L_2^2}{L_3^2}.\frac{L_4^2}{L_5^2}\ \&c.,$$

L_0, L_1, L_2, L_3, &c. being the leading coefficients of the last, the last but one, the last but two, &c. of the Bezoutian secondaries to fx and φx). Accordingly,

if $n=1$, let $\mathrm{D}=q_1=\mu_1$;

if $n=2$, let $\mathrm{D}=q_2 q_1+1=\mu_1\left\{q_2+\frac{1}{\mu_1}\right\}=\mu_1.\mu_2$;

if $n=3$, let $\mathrm{D}=q_3\{q_2 q_1+1\}+q_1=\mu_1\mu_2\left\{q_3+\frac{1}{\mu_2}\right\}=\mu_3$;

. . . . . .

and in general let $\mathrm{D}=\mu_1.\mu_2.\mu_3\ldots\mu_n$,

where $\mu_1=q_1\ \ \mu_2=q_2+\frac{1}{\mu_1}\ \ \mu_3=q_3+\frac{1}{\mu_2}\cdots\cdots\mu_n=q_n+\frac{1}{\mu_{n-1}}$.

Now suppose x to be so taken that

$$\left.\begin{array}{ll} q_1 & \text{does not lie between } +1 \text{ and } -1 \\ q_2 & \ldots\ldots\ldots +2 \text{ and } -2 \\ q_3 & \ldots\ldots\ldots +2 \text{ and } -2 \\ q_4 & \ldots\ldots\ldots +2 \text{ and } -2 \\ & \ldots\ldots\ldots \\ q_{n-1} & \ldots\ldots\ldots 2 \text{ and } -2 \\ q_n & \ldots\ldots\ldots 1 \text{ and } -1 \end{array}\right\}, \quad \ldots\ldots\ldots (\omega.)$$

where it will be observed that the excluded region lies between $+2$ and -2 for all the intermediate quotients, but between only $+1$ and -1 for the first and last quotient. Then μ_1 is positively or negatively greater than 1, therefore $\frac{1}{\mu_1}$ is a positive or negative fraction, but q_2 is positively or negatively greater than 2; therefore μ_2 will be of the same sign in q_2, and also μ_2 will be positively or negatively greater than 1; therefore $\frac{1}{\mu_2}$ will be a positive or negative fraction, but q_3 is positively or negatively greater than 2; therefore μ_3 will be of the same sign as q_3, and also μ_3 will be positively or negatively greater than 1; and proceeding in this way, we find that all values of μ_i, from $i=1$ to $i=n-1$, will be of the same sign as q_i, and positively or negatively greater than 1. Finally, $\frac{1}{\mu_{n-1}}$ will be a fraction, and therefore, since q_n is positively or negatively greater than 1, $\mu_n=q_n+\frac{1}{\mu_{n-1}}$ will have the same sign as (q_n) (but of course is not necessarily greater than 1, nor would that condition serve any purpose were it satisfied). We infer consequently, that when the conditions (ω) are satisfied, $\mu_1, \mu_2, \mu_3 \ldots \mu_n$ will respectively have the same signs as $q_1\ q_2 \ldots q_n$; and therefore $D=\mu_1.\mu_2.\mu_3 \ldots \mu_n$ has the same sign as $q_1.q_2.q_3 \ldots q_n$. Now suppose

$$q_1=a_1x+b_1 \quad q_2=a_2.x+b_2 \ldots q_n=a_n.x+b_n,$$

and solve the $2n$ equations

$$a_1x+b_1=+c_1 \quad a_2x+b_2=+c_2 \ldots a_{n-1}.x+b_{n-1}= \quad c_{n-1} \quad a_n.x+b_n= \quad c_n$$

$$a_1x+b_1=-c_1 \quad a_2x+b_2=-c_2 \ldots a_{n-1}.x+b_{n-1}=-c_{n-1} \quad a_n.x+b_n=-c_n,$$

where $c_1=1 \quad c_2=2 \quad c_3=2 \ldots\ldots c_{n-1}=2 \quad c_n=1.$

Whenever in any one of the n pairs of equations above written the coefficient of x is positive, the upper equation of the pair will bring out the greater value of x; but when the coefficient is negative the lower equation will give the greater value. Take the pair

$$a_ix+b_i=c_i$$

$$a_ix+b_i=-c_i.$$

If a_i is positive a_ix+b_i will always be positive, and greater than c_i between $x=\infty$ and $x=$ the greater of the two values of x; if a_i is negative a_ix+b_i will always be negative, and less (*i. e.* nearer to $-\infty$) than $-c_i$ for all values of x between the same limits as before. So again it will be seen in like manner, that whether a_i be positive or negative between $x=-\infty$ and $x=$ the lesser of the two values of x corresponding to the above pair of equations, a_ix+b_i will always retain the same sign, and will be greater than $+c_i$, or less than $-c_i$, according as a_i is negative or positive. If, then, we take the greatest of the *greaters* of the n pairs of values of x, *i. e.* the absolute greatest of the $2n$ values, and the least of the *lessers*, *i. e.* the absolute least of the same, say L and Λ between L and Λ, $q_1, q_2, \ldots q_n$ will each always retain an invariable sign, and will then fall without the limits $\pm c_1\ \pm c_2 \ldots\ldots \pm c_{n-1}\ \pm c_n$, so that between $+\infty$ and L and between Λ and $-\infty$, $\mu_1.\mu_2 \ldots \mu_n$, *i. e.* a constant multiple of $f(x)$, will retain the

same sign as $q_1.q_2 \ldots q_n$, *i. e.* will never change its sign from the beginning to the end of one interval, nor from the beginning to the end of the other; and consequently L and Λ will be a superior and inferior limit respectively to the real roots of fx. It will of course be observed that it is indifferent for the purposes of the foregoing theorem, whether $\frac{\varphi x}{fx}$ be expanded under the form of a proper or an improper fraction, *i. e.* whether we employ the ordinary or the Sturmian process of successive division, for changing the signs of the residues will only have the effect of changing q_i into $(\pm)q_i$, and the pair of equations $(\pm)q_i = \pm c_i$ remains the same whether the + or the − sign be prefixed to q_i. The result is, that if we form the $2n$ quantities

$$\frac{\pm 1 - b_1}{a_1}; \quad \frac{\pm 2 - b_2}{a_2}; \quad \frac{\pm 3 - b_3}{a_3}; \ \ldots \ \frac{\pm 2 - b_{n-1}}{a_{n-1}}; \ \ldots \ \frac{\pm 1 - b_n}{a_n},$$

the greatest of them will be a superior, and the least of them an inferior limit to the roots of fx*.

It may be remarked, that if the successive dividends in the course of the process be multiplied respectively by $k_1, k_2, \ldots k_n$, $\frac{\varphi x}{fx}$ will take the form

$$\frac{k_1}{q_1 +} \ \frac{k_2}{q_2 +} \ \frac{k_3}{q_3 +} \ldots \frac{k_n}{q_n};$$

and if we write $\quad a_1 x + b_1 = \pm c_1 \quad a_2 x + b_2 = \pm c_2 \ \ldots \ a_n x + b_n = \pm c_n$

and make $\quad c_1 = 1 \quad c_2 = 1 + k_2 \quad c_3 = 1 + k_3 \ \ldots \ c_n = 1 + k_n,$

the same reasoning as above will show the greatest and least of the $2n$ quantities

$$\frac{\pm 1 - b_1}{a_1}, \quad \frac{\pm (1 + k_2) - b_2}{a_2}, \ \ldots \ \frac{\pm (1 + k_n) - b_{n-1}}{a_{n-1}}; \quad \frac{\pm 1 - b_n}{a_n}$$

will be a superior and inferior limit to the roots of fx.

For greater simplicity, again, consider $k_1, k_2 \ldots k_n$ to be all equal to unity; we may make this addition to the theorem as above stated, viz. calling $L_1 \ \Lambda_1$; $L_2 \ \Lambda_2$; $\ldots$ $L_n \ \Lambda_n$ the greatest and least values of the terms contained respectively in the series marked below 1, 2, 3...n, viz.—

(1.) $\quad \frac{\pm 1 - b_1}{a_1}; \quad \frac{\pm 2 - b_2}{a_2}; \quad \frac{\pm 2 - b_3}{a_3}; \ \ldots \ \frac{\pm 2 - b_{n-1}}{a_{n-1}}; \quad \frac{\pm 1 - b_n}{a_n}$

(2.) $\quad \ldots \quad \frac{\pm 1 - b_2}{a_2}; \quad \frac{\pm 2 - b_3}{a_3}; \ \ldots \ \frac{\pm 2 - b_{n-1}}{a_{n-1}}; \quad \frac{\pm 1 - b_n}{a_n}$

(3.) $\quad \ldots \ldots \ldots \quad \frac{\pm 1 - b_3}{a_3}; \ \ldots \ \frac{\pm 2 - b_{n-1}}{a_{n-1}}; \quad \frac{\pm 1 - b_n}{a_n}$

$\quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$

$\quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$

$(n-1)$ $\quad \ldots \ldots \ldots \ldots \quad \frac{\pm 1 - b_{n-1}}{a_{n-1}}; \quad \frac{\pm 1 - b_n}{a_n}$

$(n.)$ $\quad \ldots \ldots \ldots \ldots \ldots \quad \frac{\pm 1 - b_n}{a_n},$

* For a generalization and improved form of statement of this theorem see Supplement to the present Section.

$L_1\ \Delta_1$; $L_2\ \Delta_2$; ... $L_n\ \Delta_n$ will be respectively superior and inferior limits to fx, φx and their successive residues. As a corollary, we see, of course, that L and Δ, the superior and inferior limit to the roots of the given function fx, must always lie between $+\infty$ and the greatest root, and between $-\infty$ and the least root, of the arbitrarily assumed function φx.

Art. (53.). Let us now assume somewhat more generally that φx is any number of degrees θ_1 in x lower than fx, which will cause the first quotient q_{θ_1} to be of the degree θ_1 in x; and let us further suppose that φx stands in such a relation to fx that the following quotients, $q_{\theta_2}, q_{\theta_3}, \ldots q_{\theta_\rho}$, are of the degrees $\theta_2, \theta_3 \ldots \theta_\rho$ in x ($\theta_2, \theta_3 \ldots \theta_\rho$ being supposed not necessarily units, as they would generally be, but any positive integers whatever, as may happen in consequence of one or more of the leading coefficients in any residue vanishing), then

$$\frac{\varphi x}{fx} = \frac{1}{q_{\theta_1}+}\ \frac{1}{q_{\theta_2}+}\ \frac{1}{q_{\theta_3}+}\ \ldots + \frac{1}{q_{\theta_\rho}},$$

where $\theta_1+\theta_2+\theta_3+\ldots\theta_\rho=n$, and consequently fx will be equal to the denominator of the last convergent above written, multiplied by a constant, so that we have now $c.fx=m_1.m_2\ldots m_\rho$, where

$$m_1=q_{\theta_1} \quad m_2=q_{\theta_1}+\frac{1}{m_1}\ldots m_\rho=q_{\theta_{\rho-1}}+\frac{1}{m_{\rho-1}}.$$

And as in the case previously considered, so long as

$$\begin{array}{cccccccc} & >1 & & >2 & & >2 & & >1 \\ q_{\theta_1} & \text{or} & q_{\theta_2} & \text{or} & q_{\theta_3} & \text{or} & \ldots q_{\theta_\rho} & \text{or} \\ & <-1 & & <-2 & & <-2 & & <-1 \end{array}$$

fx will have the same sign as $q_{\theta_1}.q_{\theta_2}\ldots q_{\theta_\rho}$.

Let now $\quad q_{\theta_1}=\pm c_1 \quad q_{\theta_2}=\pm c_2 \ldots q_{\theta_\rho}=\pm c_\rho,$

where $\quad c_1=1 \quad c_2=2\ldots c_{\rho-1}=2 \quad c_\rho=1.$

Consider any pair of the above equations as $q_{\theta_i}^2-c_i^2=0$.

1st. Suppose all the roots of this equation are impossible, $q_{\theta_i}^2-c_i^2$ must be positive for all values of x, and q_{θ_i} can never lie between $+c_i$ and $-c_i$; moreover, since upon the hypothesis made, $q_{\theta_i}+c_i$ and $q_{\theta_i}-c_i$ always retain the same sign, viz. that of the coefficient of the highest power of q_{θ_i}, it follows that q_{θ_i} must also always retain the same sign; for if we construct the two curves $y=q_{\theta_i}+c_i$ and $y=q_{\theta_i}-c_i$, these will both lie on the same side of the axis of x, and never cut the axis, consequently the curve $y=q_{\theta_i}$, which lies between them, must also lie on the same side as either of them, and never cut the axis.

Hence, then, if the roots of the equation are all impossible, q_{θ_i} will always retain the same sign, and will never fall within the region bounded on its two sides by $+c_i$ and $-c_i$.

2nd. Suppose the equation to have one or more possible roots, and l_i to the greatest, and λ_i the least (which of course, if there is but one possible root, will be identical). If

the leading coefficient of q_i is positive, the greatest root (l) of the equation $q_i - c = 0$ will exceed the greatest root of (l') of the equation $q_{a_i} + c_i = 0$; for between $x = \infty$ and $x = l'$, q_{a_i} must go through all values intermediate between ∞ and $-c_i$; hence there must be a quality l intermediate between l' and $+\infty$, which will make $q_{a_i} = c_i$. In like manner, if the leading coefficient of q_{a_i} is negative, it will be seen that the greatest root of $q_{a_i} + c_i = 0$ will exceed that of $q_{a_i} - c_i = 0$. Moreover, in the one case q_i will be always positive and greater than c_i, and in the other always negative, and less than c_i. In every case, therefore, between $+\infty$ and l_i, q_{a_i} retains the same sign, and does not fall within the region bounded by $+c_i$ and $-c_i$; the same thing may be shown to be true for all values of x between $-\infty$ and λ_i. Hence, then, by the same reasoning as that employed in the preceding article, we are enabled to affirm, that if we form the equation

$$(q_{a_1}^2 - 1)(q_{a_2}^2 - 4)(q_{a_3}^2 - 4) \ldots (q_{a_{\rho-1}}^2 - 4)(q_{a_\rho}^2 - 1) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\psi.)$$

its greatest root will be a superior limit, and its least root an inferior limit to the roots of the equation $fx = 0$, whatever be the value of the assumed function φx; and if the above equation (ψ.) has no real root, all the roots of fx will be imaginary.

Art. (54.). In the preceding two articles it has been supposed that all the quotients are taken integral functions of x; but the process of successive division may be so conducted as to give rise to quotients of the form

$$ax^i + bx^{i-1} + \ldots + c + \frac{d}{x} + \ldots + \frac{l}{x^{i'}}.$$

Suppose then that we have in general

$$\frac{\varphi x}{fx} = \frac{1}{q_1 +} \quad \frac{1}{q_2 +} \ldots + \frac{1}{q_n},$$

where $q_1, q_2, \ldots q_n$ are each of the general form above written (but of course i and i' being not necessarily the same for any two of the quotients), and suppose that the sum of the degrees in x of $q_1, q_2, \ldots q_n$ is $n + t$, where t is essentially (as it must be) positive. Then we shall find, as in the last article, that L and Λ being called the greatest and least roots of $(q_1^2 - 1)(q_2^2 - 4) \ldots (q_{n-1}^2 - 4)(q_n^2 - 1)$, D the denominator of the last convergent to the continued fraction above written, will never change its sign between $+\infty$ and L, nor between Λ and $-\infty$; but here we shall have

$$fx = \mathrm{K}x^t \times \mathrm{D}.$$

Hence $x^t . \mathrm{D}$ will be invariable in sign within each of these two intervals.

1st. Let t be even; then $f(x)$ will be invariable in sign, whatever L and Λ may be for each such interval.

2nd. Let t be odd; then if L is > 0 and $\Lambda < 0$, $f(x)$ cannot change its sign in either interval; but if L is < 0 or $\Lambda > 0$, fx will change its sign as x passes through zero, but will be invariable for each of the three regions contained between $+\infty$ and L, L and 0, or 0 and Λ (as the case may be), and Λ and $-\infty$; so that universally L and Λ will be a superior and inferior limit to the roots of fx, making abstraction of the roots (if any such there be in fx) whose value is zero.

Art. (55.). I shall close this section with offering (for what it is worth) a bare

suggestion as to the mode in which the theory of Intercalations may hereafter be found to admit of being extended from a system of two general functions of x, to a system of three general functions of x, y, four general functions of x, y, z, and in general to a system of s general functions of $s-1$ variables, or which is the same thing, of s homogeneous functions of s variables. In the case of two functions of x, $f(x)$ and ϕx, $fx=0$ and $\phi x=0$ may be considered to represent two systems of points in a right line; and the theory relates in this case to the relative positions of these two "Kenothemes" or point systems; and of course using x and y to denote the distances of any point in a line from two fixed points therein respectively, instead of fx and ϕx, we may employ two homogeneous functions of x and y, as $f(x, y)$ and $\varphi\ (x, y)$, to denote these two systems of points. So, similarly, if we have three functions of two variables, $f(x, y)$, $g(x, y)$, $h(x, y)$, which I shall suppose to be of the same degree, we may consider the mutual relations of the Monothemes, that is to say, the three plane curves, denoted by the equations $f(x, y)=0$, $g(x, y)=0$, $h(x, y)=0$. Now every two of these will intersect one another in a system of points, which we may call (f, g) for the intersections of f and g, (g, h) for those of $(g$ and $h)$, and (h, f) for those of h and f. If we take any two of these systems of intersections, as (f, g) and (g, h), they will both lie upon one of the given curves (g). And by reading off the two systems of points (f, g) and (g, h), arranged according to the order upon which they are disposed upon the curve g, we may, by following the course of such curve, form a scale of effective intercalations for these two systems, and in like manner for the two systems (g, h) and (h, f); (h, f) and (f, g). Now I believe that it will be found that when f, g, h represent any algebraical curves consisting of a single continuous line, either extending to infinity in both directions, or returning to itself (and I have fully satisfied myself of the truth of this for the case of ellipses), each effective scale of intercalation will contain the same number of pairs of points; if, however, the curves consist of more than one branch, as if hyperbolæ be considered, such is no longer necessarily the case; from these facts, conjoined with the light thrown upon the subject by its relation to the theory of combinants explained in the succeeding section, I am induced to infer the probability of the truth of the following law (which, for avoidance of further uncertainty, I confine to the case of functions of the same degree), viz. that if f, g, h be three homogeneous functions of x, y, and z of the same degree, and if U, V, W be any three linear functions of f, g, h, and if $U=0$, $V=0$, $W=0$ be treated as the equations to three cones, and if we form an effective scale of the intercalations of the lines of intersection of U and W, and V and W, according to the order in which they are disposed upon W (which seems to require that the lines shall be continuous, in order to admit of a fixed order of reading off the intersections of any two of them upon the third); then whatever value may have been given to the coefficients in the linear functions the number of elements remaining in any such scale will (as I conjecture) be constant, and some theory (to be discovered) for three functions analogous to that of Bezoutian residues for two functions will serve to determine the

number of the elements so remaining. And so, in like manner, but with a difficulty increasing at each step (as at the next step we should have to pass into *quasi-space* of four dimensions), a theory of intercalations may be conjectured to exist for any (n) general functions of any $(n-1)$ variables.

Development of the method of assigning a superior and inferior limit to the roots of any algebraical equation.

Art. (α.). Since the articles in the preceding part of this section on the method of discovering limits to the roots of an algebraical equation were written, the method of which the germ is therein contained has presented itself in a much more fully developed form, which I proceed to exhibit: for greater simplicity I shall suppose φx to be of $n-1$, and fx to be of n dimensions in x, and that by means of the ordinary process for common measure (except that as in STURM's theorem the sign of all the remainders are changed) $\frac{\varphi x}{fx}$ has been thrown under the form of the improper continued fraction

$$\frac{1}{q_1} - \frac{1}{q_2} - \frac{1}{q_3} - \cdots \frac{1}{q_n},$$

where $q_1 q_2 \ldots q_n$ are all restricted to signify simple linear functions of x.

Suppose the series $q_1, q_2, q_3, \ldots q_n$ to be resolved into the distinct sequences

$$q_1 q_2 \cdots q_i; \quad q_{i+1} q_{i+2} \cdots q_{i'}; \quad q_{i'+1} \cdots q_{i''} \cdots q_{(l)+1} \cdots \cdots q_n,$$

in such a manner that in each sequence as $q_{i+1} q_{i+2} \ldots q_{i'}$ the coefficients of x have all the same sign, but that in any two adjoining sequences the coefficients of x have opposite signs, so that for instance in q_i and q_{i+1} the coefficients of x are unlike, as also in $q_{i'}$ and $q_{i'+1}$; there will of course be nothing to preclude any of these sequences becoming reduced to a single term.

The first theorem is, that the greatest and least roots of the product of the cumulants

$$[q_1 q_2 \cdots q_i] \times [q_{i+1} q_{i+2} \cdots q_{i'}] \cdots \times [q_{(l)+1} q_{(l)+2} \cdots q_n]$$

are superior and inferior limits to the roots of fx. To prove this theorem I begin with premising the two following lemmas, one virtually and the other expressly contained in the Philosophical Magazine for the months of September and October of the present year*.

* Each of these two lemmata flows readily from the faculty previously adverted to engaged by every cumulant of being representable under the form of a determinant. As to the second lemma, it becomes apparent immediately when the cumulant is so represented, by separating the matrix into two rectangles and expressing the entire determinant according to a well-known rule for the decomposition of determinants as a function of the determinants belonging to these two rectangles taken separately. As to the first lemma, by reason of the cumulant $[\omega_1 \omega_2 \ldots \omega_{i-1} \omega_i \omega_{i+1}]$ being so representable, we know that when $[\omega_1 \omega_2 \ldots \omega_{i-1} \omega_i] = 0$, $[\omega_1 \omega_2 \ldots \omega_{i-1}]$ and $[\omega_1 \omega_2 \ldots \omega_{i+1}]$ must have opposite signs. Suppose, now, that the theorem is true when the number of elements in the type does not exceed i; then the roots of $[\omega_1 \omega_2 \ldots \omega_{i-1}]$, say of ψ_{i-1}, being called $k_1 k_2 \ldots k_{i-1}$, and of $[\omega_1 \omega_2 \ldots \omega_{i-1} \omega_i]$, say of ψ_i, being called $k_1 k_2 \ldots k_i$, these may be arranged in the following order of

Lemma A. The roots of the cumulant $[q_1, q_2 \ldots q_i]$, in which each element is a linear function of x, and wherein the coefficient of x for each element has the like sign, are all real, and between every two of such roots is contained a root of the cumulant $[q_1 q_2 \ldots q_{i-1}]$, and *ex converso* a root of the cumulant $[q_2 q_3 \ldots q_i]$, and (as an evident corollary) for all values of ρ and ρ' intermediate between 1 and i the greatest root of $[q_1 q_2 \ldots q_{i-1} q_i]$ will be greater, and the least root of the same will be less than the greatest and least roots respectively of $[q_\rho q_{\rho+1} \ldots q_{\rho'-1} q_{\rho'}]$.

Lemma B. For all values of the elements $q_1 q_2 \ldots q_n$, the cumulant

$$[q_1 q_2 \ldots q_{m-1} q_m q_{m+1} q_{m+2} \ldots q_n]$$
$$= [q_1 q_2 \ldots q_{m-1} q_m] \times [q_{m+1} q_{m+2} \ldots q_n]$$
$$- [q_1 q_2 \ldots q_{m-1}] \times [q_{m+2} \ldots q_n].$$

Thus *ex. gr.* the cumulant $[abcd]$, *i. e.* $abcd - ab - cd - ad + 1$,

$$= [ab] \times [cd] - [a] \times [d] = (ab-1)(cd-1) - ad,$$

and $[abcde]$, i. e. $abcde - abc - abe - ade - cde + a + c + e = [abc][ae] - [ab][e]$,

$$\text{i. e.} = (abc - a - c)(de - 1) - (ab - 1)e.$$

Art. (β.). Now suppose that $q_1 q_2 \ldots q_m q_{m+1} \ldots q_n$ are all linear functions of x, and that the coefficients of x have all one (say the positive) sign in $q_1 q_2 \ldots q_m$, and all the contrary signs in $q_{m+1} \ldots q_n$, and let L be not less than the greatest root of $[q_1 q_2 \ldots q_m]$ or of $[q_{m+1} \ldots q_n]$, and also let Λ be not greater than the least root of each of these same two cumulants; then by lemma A, L and Λ will also be respectively greater than the greatest, and less than the least roots of $[q_1 q_2 \ldots q_{m-1}]$ and of $[q_{m+2} \ldots q_n]$. Now the coefficient of the highest power of x in both $[q_1 q_2 \ldots q_m]$ and in $[q_1 q_2 \ldots q_{m-1}]$ is positive, but as to $[q_{m+1} \ldots q_n]$ and $[q_{m+2} \ldots q_n]$ is of contrary signs in the two, viz. negative in that one of those cumulants which contains an odd, and positive in that one of the two which contains an even number of elements. Hence by virtue of Lemma B, L and any quantity greater than L substituted for x will make $[q_1 q_2 \ldots q_n]$ to have always the same sign, and in like manner it may be shown that Λ and any quantity less than Λ substituted for x will also cause $[q_1 q_2 \ldots q_n]$ to retain always the same sign. Hence L and Λ are superior and inferior limits to $[q_1 q_2 \ldots q_n]$; and the same reasoning would

magnitude $k_1 h_1 k_2 h_2 k_3 \ldots k_{i-1} h_{i-1} k_i$; and if the roots of $[\omega_1 \omega_2 \ldots \omega_{i-1} \omega_i \omega_{i+1}]$, say of ψ_{i+1}, be called $l_1 l_2 \ldots . l_{i+1}$, from the fact of the leading coefficients in ψ_{i-1} and ψ_{i+1} expanded according to the powers of x having the same sign, it follows that when $x = \infty$, ψ_{i-1} and ψ_{i+1} have the same sign, but they have contrary signs when $x = k$; but ψ_{i-1} does not change its sign between $x = \infty$ and $x = k$, hence ψ_{i+1} does change its sign between $x = \infty$ and $x = k_1$, and therefore a root of ψ_{i+1} lies between ∞ and k_1; in like manner precisely it may be shown that a root of ψ_{i+1} lies between $-\infty$ and k_1; and since ψ_{i-1} changes its sign between k_1 and k_2, between k_2 and $k_3 \ldots .$ k, and between k_{i-1} and k_i, ψ_{i+1} must likewise change its sign between one and the other extremity of each of these intervals, and hence the roots $l_1 l_2 \ldots . l_{i+1}$ are intercalated between ∞, k_1, $k_2, \ldots . k_i$, $-\infty$, or which is the same thing, $k_1, k_2, \ldots . k_i$ are respectively intercalated between $l_1, l_2, \ldots l_{i+1}$; consequently, if the theorem is true up to i, it is true for $i+1$, and therefore true universally; but is manifestly true when $i = 2$, for then $x = \pm\infty$ makes $[\omega_1, \omega_2]$, *i. e.* $\omega_1 \omega_2 - 1$ positive; but $\omega_1 = 0$ makes it negative, which proves the theorem contained in Lemma A.

evidently apply if we had supposed the signs of the coefficients of x in the first partial series of elements to have been negative, and in the other series of elements to have been positive.

The greatest and least roots of $[q_1 q_2 \ldots q_m] \times [q_{m+1} \ldots q_n]$ evidently satisfy the condition to which L and Λ are subject, and may be taken in place of L and Λ respectively. They will accordingly be superior and inferior limits to the cumulant

$$[q_1 q_2 \ldots q_m \; q_{m+1} \ldots q_n].$$

Again, by virtue of theorem (B.) it may readily be shown that

$$\begin{aligned}
&[q_1 q_2 \ldots q_{m_1} q_{m_1+1} q_{m_2+2} \ldots q_{m_2} q_{m_2+1} \ldots q_n] \\
=&[q_1 q_2 \ldots q_{m_1}] \times [q_{m_1+1} q_{m_1+2} \ldots q_{m_2}] \times [q_{m_2+1} \ldots q_n] \\
-&[q_1 q_2 \ldots q_{m_1-1}] \times [q_{m_1+2} \ldots q_{m_2}] \times [q_{m_2+1} \ldots q_n] \\
-&[q_1 q_2 \ldots q_{m_1}] \times [q_{m_1+1} \ldots q_{m_2-1}] \times [q_{m_2+2} \ldots q_n] \\
+&[q_1 q_2 \ldots q_{m_1-1}] \times [q_{m_1+2} \ldots q_{m_2-1}] \times [q_{m_2+2} \ldots q_n];
\end{aligned}$$

and hence if $q_1 q_2 \ldots q_n$ are all linear functions of x in which the coefficients of x have all the same algebraical sign in any one (taken *per se*) of the three series

$$q_1 q_2 \ldots q_{m_1}; \quad q_{m_1+1} \ldots q_{m_2}; \quad q_{m_2+1} \ldots q_n;$$

but so that this sign changes in passing from one series to another, it is easily seen, by the same reasoning as in the preceding case, that the two positive and two negative products on the right-hand side of the equation all give the same sign to the coefficient of the highest power of x, and consequently that if L and Λ be superior and inferior limits to

$$[q_1 \ldots q_{m_1}], \quad [q_{m_1+1} \ldots q_{m_2}], \quad [q_{m_2+1} \ldots q_n],$$

and consequently by Lemma A, to

$$[q_1 q_2 \ldots q_{m_1-1}], \quad [q_{m_1+2} \ldots q_{m_2}], \quad [q_{m_1+1} \ldots q_{m_2-1}], \quad [q_{m_1+2} \ldots q_{m_2-1}], \quad \text{and to} \quad [q_{m_2+2} \ldots q_n],$$

L or Λ substituted for x will cause $[q_1 q_2 \ldots q_n]$ to retain always the same sign, and will consequently be superior and inferior limits thereto; and so in general; whence it follows, returning to the theorem to be demonstrated, that the greatest and least roots of

$$[q_1 q_2 \ldots q_i] \times [q_{i+1} q_{i+2} \ldots q_\ell] \times \ldots \times [q_{\lambda+1} \ldots q_n],$$

will be superior and inferior limits to the cumulant $[q_1 q_2 \ldots q_n]$, *i. e.* to C.fx*, and therefore to fx, as was to be proved.

* If $\frac{\phi x}{fx}$ expanded as a continued fraction by means of the common measure process gives rise to the quotients $q_1, q_2, \ldots q_n$, and if $L_1, L_2, \ldots L_{n-1}, L_n$ be the leading coefficients of the successive simplified residues, (L_n being, in fact, the final simplified residue, *i. e.* the resultant to $\phi x, fx$), we must have $\phi x = C[q_1, q_2 \ldots q_n]$ $fx = C[q_1, q_2 \ldots q_n]$, where (supposing ϕx to be of $n-1$, and fx of n dimensions in x),

$$C = \frac{1}{L_n} \left\{ \begin{matrix} L_n^2 \cdot L_{n-2}^3 \cdot L_{n-4}^3 \text{ \&c.} \\ L_{n-1}^2 \cdot L_{n-3}^3 \cdot L_{n-5}^3 \text{ \&c.} \end{matrix} \right\}.$$

Art. (γ.). The second theorem is the following: if $q_1, q_2, \ldots q_n$ be linear functions of x, say a_1x+b_1, $a_2x+b_2 \ldots a_nx+b_n$, in which the coefficients of x have all the same sign, and if we take the quantities $\mu_1, \mu_2, \ldots \mu_{n-1}$, all having the same sign as $a_1, a_2 \ldots a_n$, but otherwise arbitrary, and make

$$k_1=\mu_1 \quad k_2=\mu_2+\frac{1}{\mu_1} \quad k_3=\mu_3+\frac{1}{\mu_2} \cdots k_{n-1}=\mu_{n-2}+\frac{1}{\mu_{n-2}} \quad k_n=\frac{1}{\mu_n},$$

then the greatest of the quantities

$$\frac{k_1-b_1}{a_1}, \quad \frac{k_2+b_2}{a_2}, \quad \ldots \frac{k_n-b_n}{a_n},$$

say L, is a superior limit, and the least of the quantities

$$\frac{-k_1-b_1}{a_1}, \quad \frac{-k_2-b_2}{a_2}, \ldots \frac{-k_n-b_n}{a_n},$$

say Λ, is an inferior limit to the roots of fx.

L and any value greater than L substituted for x will evidently make q_1-k_1; q_2-k_2; ...; q_n-k_n, all of them positive.

Hence when $x=$ or $>$L q_1 is positive and $>\mu_1$ and

$$q_2-\frac{1}{q_1}>k_2-\frac{1}{\mu_1}>\mu_2+\frac{1}{\mu_1}-\frac{1}{\mu_1}, \text{ } i.\ e. \text{ is positive, and } >\mu_2,$$

$$q_3-\frac{1}{q_2-}\ \frac{1}{q_1}>k_3-\frac{1}{\mu_2}>\mu_3+\frac{1}{\mu_2}-\frac{1}{\mu_2}, \text{ } i.\ e. \text{ is positive, and } >\mu_3,$$

. . . . . .

. . . . . .

and $q_n-\frac{1}{q_{n-1}-}\ \frac{1}{q_{n-2}}\cdots\frac{1}{q_1}>\frac{1}{\mu_{n-1}}-\frac{1}{\mu_{n-1}}$, *i. e.* is positive,

and consequently the cumulant $[q_1q_2q_3 \ldots q_n]$, which

$$=q_1\times\left(q_2-\frac{1}{q_1}\right)\times\left(q_3-\frac{1}{q_2-}\ \frac{1}{q_1}\right)\times \&c.,$$

remains of a constant sign when L and any quantity greater than L is substituted for x. Hence L is a superior limit. In like manner Λ and any quantity less than Λ will evidently make q_1+k_1, q_2+k_2; ... q_n+k_n all of them negative, so that when $x=$ or $<\Lambda$ q_1 is negative, and $<-\mu_1$

$$q_2-\frac{1}{q_1}<k_2-\frac{1}{\mu} \text{ is negative, and } <-\mu_2,$$

$$q_3-\frac{1}{q_2}<k_3-\frac{1}{\mu_2} \text{ is negative, and } <-\mu_3,$$

. . . . . .

. . . . . .

. . . . . .

and $q_n-\frac{1}{q_{n-1}-}\ \frac{1}{q_{n-2}}\cdots\frac{1}{q_1}<\frac{1}{\mu_{n-1}}-\frac{1}{\mu_{n-1}}$ is negative.

So that $[q_1\, q_2 \ldots q_n]$ for all values of x less than Λ will preserve an invariable sign, and consequently Λ is an inferior limit to fx.

Art. (δ.). It may be remarked that the quantities

$$\mu_1;\ \mu_2+\frac{1}{\mu_1};\ \mu_3+\frac{1}{\mu_2};\ \ldots\ \mu_{n-2}+\frac{1}{u_{n-3}};\ \mu_{n-1}+\frac{1}{\mu_{n-2}};\ \frac{1}{\mu_{n-1}}$$

may be derived successively from one another, according to the same law, from whichever end of the series we begin.

If we take any two consecutive terms as

$$\mu_i+\frac{1}{\mu_{i-1}};\ \mu_{i+1}+\frac{1}{\mu_i},$$

the effect of diminishing μ_i is to decrease the first of these two terms, and *pro tanto*, to tend to deduce the limit; but on the other hand, $\frac{1}{\mu_i}$ being increased, there is brought into play an opposite tendency, which operates *pro tanto* to increase the value of the limit.

Art. (ε.). It is of importance to remark, that by a right selection of the system of quantities $\mu_1\, \mu_2 \ldots \mu_{n-1}$, which enter into the composition of $k_1\, k_2 \ldots k_n$, L may be made to coincide with the greatest root of $[q_1\, q_2 \ldots q_n]$; and so in like manner by a right selection of another system of these quantities, whereby to form $k_1\, k_2 \ldots k_n$, Λ may be made to coincide with the least root of the same. Thus let $\mu_1\, \mu_2 \ldots \mu_{n-1}$ be so chosen, that

$$q_1-k_1=0 \quad q_2-k_2=0 \ldots q_n-k_n=0$$

are all satisfied by the same value of x.

Then $\quad q_1=\mu_1 \quad q_2=\mu_2+\frac{1}{\mu_1} \quad q_3=\mu_3+\frac{1}{\mu_2} \cdots q_n=\frac{1}{\mu_{n-1}}$

exist simultaneously.

Hence $\quad \mu_2=q_2-\frac{1}{q_1} \quad \mu_3=q_3-\frac{1}{\mu_2}=q_3-\frac{1}{q_2-\frac{1}{q_1}}$

$$\mu_{n-1}=q_{n-1}-\frac{1}{q_{n-2}-\frac{1}{q_{n-3}-\ldots-\frac{1}{q_1}}}$$

$$q_n=\frac{1}{q_{n-1}-\frac{1}{q_{n-2}-\ldots-\frac{1}{q_1}}},$$

which is satisfied by making $[q_n, q_{n-1}, q_{n-2} \ldots q_1]=0$.

It remains then only to show that the greatest root of x in this equation substituted for x in $q_1, q_2, \ldots q_n$ will make $\mu_1\, \mu_2 \ldots \mu_{n-1}$ all of one sign, and that the least root of x similarly substituted, will also make them all of one, but a contrary sign, which may be proved as follows.

We have

$\mu_1=q_1\ \mu_2=[q_1, q_2] \div q_1\ \mu_3=[q_1\, q_2\, q_3] \div [q_1, q_2]$ &c. $\mu_{n-1}=[q_1\, q_2 \ldots q_{n-1}] \div [q_1, q_2 \ldots q_{n-2}]$; and by Lemma B the superior limit to $[q_1\, q_2 \ldots q_n]$ will be a superior limit also to $q_1, q_2\, q_3, \ldots, q_{n-2}$, and to $\quad [q_1, q_2], [q_1\, q_2\, q_3] \ldots, [q_1\, q_2, \ldots, q_{n-1}]$.

Consequently this superior limit will make $\mu_1, \mu_2 \ldots \mu_{n-1}$ have all the same sign as that of the coefficients of x in $q_1, q_2, \ldots q_n$. And in like manner, the inferior limit to $[q_1, q_2, \ldots q_n]$ will cause $\mu_1, \mu_2, \ldots \mu_{n-1}$ to have all the contrary sign to that of these coefficients.

Thus then we see that when the coefficients of x in the partial quotients to $\frac{\varphi x}{fx}$ expressed as an improper continued fraction form a single series of continuations of signs, by a right choice of the arbitrary constants $\mu_1, \mu_2, \ldots \mu_{n-1}$ the superior or inferior limit given by this new method may severally and separately be made to coincide with the greatest and least real root, or each in turn with the sole real root of fx, if there be but one.

Art. (ζ.). The general method of enclosing the roots of fx within limits is founded upon the combination of the two theorems above demonstrated. An arbitrary function φx one degree in x below, fx being assumed, and by aid of the auxiliary function $\varphi x, fx$ being thrown under the form

$$C[q_1, q_2, \ldots q_i, q'_1, q'_2, \ldots q'_{i'}, q''_1 \ldots\ldots (q)_1, (q)_2 \ldots, (q)_\varepsilon],$$

in which the coefficient of x is supposed to change sign in the passage from q_i to q'_1, from $q_{i'}$ to q''_1, &c., a superior limit is found to each of the cumulants

$$[q_1 q_2 \ldots q_i], \ [q'_1 q'_2 \ldots q_{i'}], \ \ldots \ [(q)_1 (q)_2 \ldots (q)_{(\omega)}],$$

taken separately, by means of the second theorem, and then by virtue of the first theorem the greatest of these superior limits is a superior limit to the cumulant

$$[q_1 q_2 \ldots q_i \ldots (q)_1 \ldots (q)_{(\omega)}],$$

and consequently to fx, and so *mutatis mutandis* the least of the inferior limits of the same partial cumulants is an inferior limit to the total cumulant

$$[q_1 q_2 \ldots q_i \ldots (q)_1 (q)_2 \ldots (q)_{(\omega)}].$$

Art. (η.). When all the roots of fx are real, if φx be so assumed that all its roots are intercalated between those of fx, the partial quotients to $\frac{\varphi x}{fx}$ will form but one single series. In order that φx may fulfill this condition, it is necessary that the coefficients of φx shall be subject to certain conditions of inequality, not necessary here to be investigated; but no conditions of equality, *i. e.* no equations between the coefficients of φx, are introduced by this condition; or in other words, the coefficients* of φx, the auxiliary function, are independent and arbitrary within limits; and we have shown that in this case the auxiliary constants $\mu_1 \mu_2 \ldots \mu_{n-1}$ may be so determined that the limits may be made to come separately and respectively into contact with the two extreme roots. When all the roots of fx are not real, the quotients (however φx is chosen) can no longer be made to form a single series. It still however remains true, that, by a due choice of the auxiliary function followed by a due choice of the

* It need scarcely be stated that $f'x$ is the simplest form of φx, which satisfies the condition in question.

auxiliary constants, this coincidence may be brought about, so long as there is a single real root in fx.

It is rather important to demonstrate this universal possibility of effecting a coincidence of the limits to the roots with the extreme roots themselves, because it is the most striking feature which distinguishes the method of limitation here developed from all others previously brought to light.

Art. (θ.). Before entering upon this demonstration I may make the passing remark, that every method of root-limitation is implicitly a method of root-approximation.

For instance, let e be any given quantity between which and $+\infty$ it is known that a root of fx lies. Then if we write $x=e+\frac{1}{y}$, and form the equation $y^n f\left(e+\frac{1}{y}\right)=0$, and find L a superior limit to y, it is clear that $e+\frac{1}{\mathrm{L}}$ will lie between e and the root of fx say E, next superior to e. Again, making $x=e+\frac{1}{\mathrm{L}}+\frac{1}{y'}$, and finding a superior limit L′ to y', we shall have $e+\frac{1}{\mathrm{L}}+\frac{1}{\mathrm{L}'}$ still nearer to E than $e+\frac{1}{\mathrm{L}}$ was; and so we may proceed advancing nearer and nearer, and always from the same side towards E at each step, and finally obtain E under the form $e+\frac{1}{\mathrm{L}}+\frac{1}{\mathrm{L}'}+\frac{1}{\mathrm{L}''}+$ &c. And in like manner calling E, the root next below e, we may find $\mathrm{E}_1=e-\frac{1}{\Lambda}-\frac{1}{\Lambda'}-\frac{1}{\Lambda''}$, &c.

Art. (ι.). In establishing the theorem of coincidence above adverted to, the following notation will be found very advantageous. Let Ω denote a Type of any number of Elements, as $q_1\ q_2 \ldots q_{i-1}\ q_i$, and let Ω' denote this same type when the last element, and $'\Omega$ the same type when the first element is cut off, and $'\Omega'$ the same type when both extremes are cut off, so that the apocopated type Ω' will mean $[q_1\ q_2 \ldots q_{i-1}]$; the apocopated type $'\Omega$ will mean $[q_2\ q_3 \ldots q_i]$, and the doubly apocopated type $'\Omega'$ will mean $[q_2\ q_3 \ldots q_{i-1}]$.

If now a type Ω be made up of the types $\Omega_1\ \Omega_2 \ldots \Omega_i$ put in apposition, and if we use in general $[\Omega]$ to denote the cumulant corresponding to the type Ω, there will be a very simple law* connecting $[\Omega]$ with

$$
\begin{array}{c}
[\Omega_1][\Omega_2][\Omega_3]\ldots[\Omega_{i-2}][\Omega_{i-1}][\Omega_i] \\
[\Omega_1'][\Omega_2'][\Omega_3']\ldots[\Omega_{i-2}'][\Omega_{i-1}'] \\
['\Omega_2]['\Omega_3]\ldots['\Omega_{i-2}]['\Omega_{i-1}]['\Omega_i] \\
['\Omega_2']['\Omega_3']\ldots['\Omega_{i-2}']['\Omega_{i-1}'].
\end{array}
$$

This law will be seen to be obviously deducible by successive steps of expansion

* The cumulant corresponding to any portion or fragment of a type may be said to be a partial cumulant to the entire type, and a type whose elements are constituted out of the elements of two or more types placed in juxtaposition may be said to be the aggregate of these types; the law given in the text above may then be said to have for its object the expansion of the complete cumulant to any type in terms of complete and partial cumulants to the types of which the given type is the aggregate.

from the fundamental theorem given in Lemma (B.) art. (*i.*), for the case of $\Omega=\Omega_1\Omega_2$, and will be best understood by showing its operation in a few simple cases.

Thus let $\Omega=\Omega_1\,\Omega_2$*.

Then $$[\Omega]=[\Omega_1]\times[\Omega_2]-[\Omega_1']\times[\Omega_2].$$

Let $\Omega=\Omega_1\,\Omega_2\,\Omega_3$.

Then $$[\Omega]=[\Omega_1]\times[\Omega_2]\times[\Omega_3]$$
$$-[\Omega_1']\times[\Omega_2]\times[\Omega_3]-[\Omega_1]\times[\Omega_2']\times[\Omega_3]$$
$$+[\Omega_1']\times[\Omega_2']\times[\Omega_3].$$

Let $\Omega=\Omega_1\,\Omega_2\,\Omega_3\,\Omega_4$.

Then $$[\Omega]=[\Omega_1]\times[\Omega_2]\times[\Omega_3]\times[\Omega_4]$$
$$-[\Omega_1']\times[\Omega_2]\times[\Omega_3]\times[\Omega_4]-[\Omega_1]\times[\Omega_2']\times[\Omega_3]\times[\Omega_4]-[\Omega_1]\times[\Omega_2]\times[\Omega_3']\times[\Omega_4]+$$
$$+[\Omega_1']\times[\Omega_2']\times[\Omega_3]\times[\Omega_4]-[\Omega_1']\times[\Omega_2]\times[\Omega_3']\times[\Omega_4]-[\Omega_1]\times[\Omega_2']\times[\Omega_3']\times[\Omega_4]$$
$$-[\Omega_1']\times[\Omega_2']\times[\Omega_3']\times[\Omega_4],$$

and so in general if $\Omega=\Omega_1\Omega_2\ldots\Omega_i$, $[\Omega]$ may be expanded under the form of the sum of 2^{i-1} products separable into i alternately positive and negative groups containing respectively 1, $(i-1)$, $(i-1)\frac{i-2}{2}$, $\ldots(i-1)$, 1 products.

Art. (*κ.*). In every one of the above groups forming a product the accents enter in pairs and between contiguous factors, it being a condition that if any Ω have an accent on the right the next Ω must have one on the left, and if it have one on the left the preceding Ω must have an accent on the right, and the number of pairs of accents goes on increasing in each group from 0 to $i-1$. This rule serves completely to define the development in question‡.

* The sign of equality is employed here to denote the relation between a concrete whole and the aggregate of its parts.

† The number of distinct factors entering into these products, taken collectively, is evidently $i+2(i-1)+(i-2)$, i. e. $4(i-1)$.

‡ When each partial type Ω consists of a single element, every doubly accented Ω will vanish, and every singly accented Ω will become unity; hence we may derive the rule for the expansion of the cumulant $[a_1\,a_2\,a_3,\ldots a_i]$ in terms of $a_1\,a_2\ldots a_i$, which will accordingly consist of

$$a_1\,.\,a_2\,.\,a_3\ldots a_i-\Sigma\frac{1}{a_e\,.\,a_{e+1}}(a_1\,.\,a_2\ldots a_i)+\Sigma\frac{1}{a_e\,.\,a_{e+1}\times a_f\,.\,a_{f+1}}(a_1\,.\,a_2\ldots\ldots a_i)\mp\&c.,$$

the indices e and f, $e+1$ and f, &c. being understood to be all distinct integers (which agrees with the known rule for the expression of the denominator of a continued fraction in terms of the quotients). The number of terms in this expansion, in consequence of the vanishing of the quantities affected with a double accent, reduces from 2^{i-1} down to the ith term in the series commencing with 1, 2, 3, &c. defined by the equation $u_{i+1}=u_i+u_{i-1}$,

$$i.\,e.\quad\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{i+1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{i+1};$$

the number, therefore, of products in which double accents occur in the general expansion of $[\omega_1\,\omega_2\ldots\omega_i]$ is

$$2^{i-1}-\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{i+1}+\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{i+1}.$$

For greater brevity let $[\Omega_s][\Omega_t][\Omega_u][\Omega_v]$ be denoted respectively by $\omega_s, \omega'_s, {}'\omega_s, {}'\omega'_s$; then when the type Ω_s consists of a single element,

$$\omega'_s=1 \quad {}'\omega_s=1 \quad {}'\omega'_s=0.$$

It should be observed that the two equations $\omega_s=0$ $\omega'_s=0$ cannot exist simultaneously, for if Ω_s represent $q_1 q_2 \ldots q_i$,

$$\omega_s=q_i\omega'_s-\omega''_s \quad \omega'_s=q_{i-1}\omega''_s-\omega'''_s, \text{ \&c.},$$

so that if $\omega_s=0$ and $\omega'_s=0$, we have $\omega''_s=0$, $\omega'''_s=0$, &c., and thus, finally, $-1=0$, which is absurd.

Now, if we suppose $\Omega_1 \Omega_2 \ldots \Omega_s$ to be types every element in each of which is a linear function of x, the coefficients of x in these elements being positive in Ω_1, negative in Ω_2, and so on alternately, and Ω is the aggregate of $\Omega_1 \Omega_2 \ldots \Omega_s$, it may easily be made out that each term in the development of ω in terms of $\omega_1, \omega'_1, {}'\omega_1, {}'\omega'_1$; $\omega_2, \omega'_2, {}'\omega_2, {}'\omega'_2$; &c. will have the same sign when we give to x a value which is a superior limit, or an inferior limit to the roots of each of the cumulants $\omega_1, \omega_2, \ldots \omega_s$, and consequently to those of the cumulants $\omega'_1, \omega'_2, \ldots \omega'_s$; ${}'\omega_1, {}'\omega_2, \ldots {}'\omega_s$; ${}'\omega'_1, {}'\omega'_2, \ldots {}'\omega'_s$; the products affected with positive signs being all positive or negative in themselves, and those affected with negative signs being reversely all negative, or all positive.

Thus, *ex. gr.* if

$$\Omega=\Omega_1\Omega_2$$

$$\omega=\omega_1.\omega_2-\omega'_1.{}'\omega_2,$$

and the sign of the leading coefficient in ${}'\omega_2$ will be the contrary of that in ω_2, but ω_1 and ω'_1 have both the same positive sign; so again if $\Omega=\Omega_1\Omega_2\Omega_3$,

$$\omega=\omega_1.\omega_2.\omega_3-\omega'_1.{}'\omega_2.\omega_3-\omega_1.\omega'_2.{}'\omega_3+\omega'_1.{}'\omega_2.{}'\omega_3,$$

where the leading coefficients in ω_2 and ${}'\omega_2$ have contrary signs, as have also those in ω_3 and ω'_3 between ω_2 and ${}'\omega'_2$ have the same sign; and of course the leading coefficients in $\omega_1, \omega_2, \omega'_1, {}'\omega_3$ have all the same sign, they being all positive, and so in general. But the superior limit to the roots of any integral algebraical function of x substituted in place of x causes the signs of the resulting values of the functions to coincide with the signs of the leading coefficients, so that in the example last above given, L a superior limit to all the factors in the several products in the equation substituted for x will make $\omega_1.\omega_2.\omega_3$, $-\omega'_1.{}'\omega_2.\omega_3$, $-\omega_1.\omega'_2.{}'\omega_3$, $\omega_1\,{}'\omega'_2.{}'\omega_3$ to have all the same sign. The like will be true of Λ the inferior limit; for if Ω_1, Ω_2, Ω_3 contain respectively n_1, n_2, n_3 elements, the values of the four products last above written, when $x=-\infty$, will be to the values of the same when $x=+\infty$ in the respective ratios of

$$(-)^{n_1+n_2+n_3}:1\,; \quad (-)^{n_1+n_2+n_3-2}:1\,; \quad (-)^{n_1+n_2+n_3-2}:1\,; \quad (-)^{n_1+n_2+n_3-4}:1,$$

and so in general. Hence we deduce the theorem, that if the total type Ω represent the aggregate in apposition of the partial orders $\Omega_1\Omega_2\ldots\Omega_s$ (the elements being understood to be linear functions of x, which are subject to the law of alternation in the signs of the coefficients of x in passing from one partial type to another), no superior

limit to $\omega_1, \omega_2, \ldots \omega_e$ can make ω vanish unless each separate product in the expansion of ω in terms of $\omega_1, \omega_2, \ldots \omega_e$ and the appurtenant apocopated cumulants vanish separately.

Art. (λ.). From the above theorem we may deduce the following law, viz. that if the roots of $\omega_1, \omega_2, \ldots \omega_e$ be supposed to be arranged in order of magnitude, and λ to be that one of them which is nearest to $+\infty$ or to $-\infty$, then if e is even it is impossible for λ to be a root of ω. Thus suppose $e=2$, and consequently $\omega=\omega_1.\omega_2-\omega_1'.\omega_2'$; if λ be a root of ω_1 and one of the two extremes of the roots of ω_1, ω_2 put in order of magnitude, λ cannot be a root of ω_2, for the roots of ω_2 are confined between the roots of ω_2; but if λ make ω and ω_1 each vanish, we must have $\omega_1'.\omega_2=0$, hence $\omega_1'=0$ as well as $\omega_1=0$, which is impossible. In like manner if a root of ω_2 were the extreme root, the same impossibility could be in like manner established.

Again, suppose $e=4$, so that

$$\omega=\omega_1.\omega_2.\omega_3.\omega_4\left\{1-\frac{\omega_1'.\omega_2'}{\omega_1.\omega_2}-\frac{\omega_2'.\omega_3'}{\omega_2.\omega_3}-\frac{\omega_3'.\omega_4'}{\omega_3.\omega_4}+\frac{\omega_1'.\omega_2'.\omega_3'}{\omega_1.\omega_2.\omega_3}+\frac{\omega_1'.\omega_2'.\omega_3'.\omega_4'}{\omega_1.\omega_2.\omega_3.\omega_4}+\frac{\omega_1.\omega_3'.\omega_3'.\omega_4'}{\omega_2.\omega_3.\omega_4}-\frac{\omega_1'.\omega_2'.\omega_3'.\omega_4'}{\omega_1.\omega_2.\omega_3.\omega_4}\right\}.$$

Let λ continue to denote one or the other extreme of the roots of $\omega_1\ \omega_2\ \omega_3\ \omega_4$. We must in each case, if λ makes $\omega=0$, have

$$\omega_1.\omega_2.\omega_3.\omega_4'=0\ ;\ \omega_1'.\omega_2.\omega_3.\omega_4=0\ ;\ \omega_1.\omega_2'.\omega_3'.\omega_4=0\ ;\ \omega_1.\omega_2.\omega_3'.\omega_4=0\ ;$$
$$\omega_1'.\omega_2'.\omega_3'=0\ ;\ \omega_1'.\omega_2.\omega_3'.\omega_4=0\ ;\ \omega_1.\omega_2'.\omega_3'.\omega_4=0\ ;\ \omega_1'.\omega_2'.\omega_3'.\omega_4=0.$$

Now suppose that λ is a root of ω_1, then the equations remaining to be satisfied are

$$\omega_1'.\omega_2.\omega_3.\omega_4=0\ ;\ \omega_1'.\omega_2'.\omega_3=0\ ;\ \omega_1'.\omega_2.\omega_3'.\omega_4=0\ ;\ \omega_1'.\omega_2'.\omega_3'.\omega_4=0.$$

Since ω_1 and ω_1' cannot both be zero together, λ cannot make ω_1' or $'\omega_1$ zero; and because λ is an extreme to the roots of $\omega_2, \omega_3, \omega_4$, λ cannot make ω_2' or $'\omega_2$ or ω_3' or $'\omega_3$ or ω_4' or $'\omega_4$ zero, so that in fact when $x=\lambda$ none of the singly accented quantities ω can be zero. As regards the doubly accented quantities ω, the same thing cannot be affirmed, because if any Ω contains only one element the corresponding value of ω with a double accent vanishes spontaneously. Again, any of the unaccented quantities ω may vanish, because we may suppose any of these to have an extreme root λ. Consequently the first, second and fourth of the equations remaining to be satisfied, might be satisfied on making the necessary suppositions as to the form of the quantities ω and the values of the extreme roots; but the third remaining equation $\omega_1'.\omega_2.\omega_3'.\omega_4=0$, in which only singly accented quantities ω occur, remains incapable of being satisfied on any supposition whatever. And the same thing would be true if we suppose λ to be a root of any other ω instead of ω_1. Hence λ cannot make $\omega=0$ when $e=4$.

In like manner, if e be any even number $2s$, there will be an equation

$$\omega_1'.\omega_2.\omega_3'.\omega_4.\omega_5'.\omega_6\ldots\omega_{2s-1}'.\omega_{2s}=0$$

to be satisfied by that value (if it exist) of x which, besides being an extreme (on either side) of the roots of $\omega_1, \omega_2, \ldots \omega_{2s}$ arranged in order of magnitude, also makes $\omega=0$. But as such equation cannot be satisfied, neither extreme root of the roots of

$\omega_1.\omega_3\ldots\omega_{2\varepsilon}$ can be a root of ω, as was to be proved. Consequently, unless φx is so assumed that the number of changes of sign in the coefficients of x in the quotients resulting from $\frac{\varphi x}{fx}$ expanded as an improper continued fraction is even (for if the *changes* from sequence to sequence are odd the number of *sequences* themselves is even), the method of limitation in the text cannot give the means of drawing either limit indefinitely near to one or the other extreme roots of fx.

Art. (μ.). It now remains to prove the converse, and to show, 1st, that when the number of changes is even, *i. e.* the number of sequences odd, this coincidence can always be effected; and 2ndly, that it is always possible when fx has one or more real roots, so to assume φx that the number of sequences shall be odd.

The first part of the proposition is easily proved. Thus suppose $e=3$, so that

$$\omega=\omega_1.\omega_2.\omega_3-\omega'_1.'\omega_2.\omega_3-\omega_1.\omega'_2.'\omega_3+\omega'_1.'\omega'_2.'\omega_3.$$

If we suppose λ either extreme of the scale formed by writing in order of magnitude, the roots of ω_1, ω_2, ω_3 to be a root common to ω_1 and to ω_3 and if $'\omega'_2=0$, which last equation may be satisfied by supposing the type Ω_2 to consist of a single element, the separate equations

$$\omega_1.\omega_2.\omega_3=0\quad \omega'_1.'\omega_2.\omega_3=0\quad \omega_1.\omega'_2.'\omega_3=0\quad \omega'_1.'\omega'_2.'\omega_3=0$$

will all be satisfied; and so in general it may be shown without difficulty that if $e=2\varepsilon+1$, and if λ be a root common to $\omega_1=0$ $\omega_3=0$ $\omega_5=0\ldots\omega_{2\varepsilon+1}=0$, and if $\omega_2, \omega_4, \ldots \omega_{2\varepsilon}$ be all *simple linear functions* of x, so that consequently $'\omega'_2=0$ $'\omega'_4=0\ldots'\omega'_{2\varepsilon}=0$, each separate term in the development of ω will vanish singly and separately, and consequently λ will be a root of ω: for since λ makes $\omega_1=0$ $\omega_3=0\ldots\omega_{2k+1}=0$, every product in the developed form ω_r in which $\omega_1, \omega_3, \ldots\omega_{2k+1}$ do not each bear at least one accent, will vanish; and if we consider any product in which $\omega_1, \omega_3\ldots\omega_{2k+1}$ are all accented, if in any two of these immediately following one after the other as ω_{2k-1}, ω_{2k+1}, an accent falls to the right of the first, and to the left of the second, the intervening term ω_{2k} will bear a double accent, and will therefore vanish, since Ω_{2k} is supposed to be a linear function of x; but it is impossible when every ω is accented to prevent two accents of contiguous odd terms in any such product, from falling to the right of the left, and to the left of the right, term of the two, since the contrary would imply that all the accents would fall to the right, or all to the left, which, as above remarked, is impossible, on accoun of the two extreme terms being only simply accentable, *i. e.* ω_1 only to the right, and ω_{2k+1} only to the left. Hence, when x substituted for λ makes $\omega_1\ \omega_3\ldots\omega_{2k+1}$ all vanish, and when $\omega_2, \omega_4, \ldots \omega_{2k}$ are all linear functions of x, $x=\lambda$ will be a root of ω.

Art. (ν.). I believe that the remaining part of the proposition may be rigorously demonstrated, viz. that when any of the roots of fx are real, and the number of odd integers not exceeding the index of the degree of fx is m, and the number of imaginary pairs of roots in fx is μ, φx may be so assumed that the quotients to $\frac{\varphi x}{fx}$ expanded

under the form of an improper continued fraction, may be made to take the form Ω_1; Ω_2; Ω_3; Ω_4; ...; Ω_{2i+1}, where Ω_2; Ω_4; ... Ω_{2i} are linear functions of x, and i is any number assumed at will, not less than μ, and of course not greater than m; and where ω_1; ω_3; ... ω_{2i+1} will have in common a root λ, which may be made at will the greatest or the least root of $\omega_1.\omega_3.\omega_5.\omega_{2i+1}$; the investigation, however, according to the present light which I possess on the subject, appears complicated and tedious, and therefore, in order that the press, which is waiting for the completion of these supplemental articles, may not be kept standing, must be adjourned to some future occasion. For the present I content myself with showing the truth of the law for the simple case where fx is a cubic function of x.

1st. If $\frac{\varphi x}{fx}$ gives rise to a single sequence of quotients Ω, we know, from the theory of intercalations, that it is necessary that all the roots of fx shall be real, and in order that when this is the case the quotients may form a single sequence Ω, it is only necessary so to assume φx, that its roots may be intermediate between those of fx.

2nd. If the roots of fx are not all real, or if they are all real, but do not compose the roots of fx intercalated between them, and if for greater brevity of ratiocination we stipulate that φx shall have its leading coefficients of the same sign as that of the leading coefficient of fx, the leading coefficients of the three quotients will either bear the respective signs $++-$, or the respective signs $+-+$, or the respective signs $+--$; in the first and last of these cases there would be two sequences, and therefore, by what has been shown above, the method of limitation of the text could not give a limit coincident with a root. Let us then look to the remaining case, and inquire whether, and how, φx may be assumed so that fx shall become representable to a constant factor *près* by the cumulant $[p(x-a), -q(x-\beta), r(x-a)]$, where p, q, r are all positive, and a is a root of fx.

Let this cumulant be called hfx.

Nothing in point of generality will be lost if we suppose the leading coefficient of hfx to be -1. We then have

$$hfx=[p(x-a),\ -q(x-\beta);\ r(x-a)]$$
$$=-pqr(x-a)^2(x-b)-(p+r)(x-a)$$

and writing $\frac{hfx}{x-a}=x^2+Bx+C$ and making $x=a$, we find from the above identity that

$$p+r=a^2+Ba+C,\ i.e.\ p=a^2+Ba+C-r,$$

and
$$pqr(x-\beta)=x+a+B,$$

hence
$$\beta+a+B=0,\ i.e.\ \beta=-B-a,$$

and
$$pqr=1,\ \text{and}\ \therefore\ qr=\frac{1}{p}=\frac{1}{a^2+Ba+C-r}.$$

Hence if φx be so assumed that the quotients to $\frac{\varphi x}{fx}$ are $p(x-a)$; $-q(x-\beta)$; $r(x-a)$,

we have

$$h\varphi x=[-q(x-\beta),\ r(x-a)]=-qr(x+\mathrm{B}+a)(x-a)-1\ ,$$

$$=-qr(x^2+\mathrm{B}x-a^2-a\mathrm{B})-1=-\frac{1}{p}\{x^2+\mathrm{B}x-a^2-a\mathrm{B}+p\}.$$

Hence $\varphi(x)$ is of the form

$$m(x^2+\mathrm{B}x-a^2-a\mathrm{B}+(a^2+a\mathrm{B}+\mathrm{C}-r))=m(x^2+\mathrm{B}x+\mathrm{C}-r).$$

If we call the three roots of fx, a, b, c respectively, we have

$$q=\frac{1}{r(a^2+\mathrm{B}a+\mathrm{C}-r)}=\frac{1}{r((a-b)(a-c)+r)}\ ;$$

and since q and r are both to be positive, we see that (a) must be taken the greatest or least of the three roots if they are all real, so that $a^2+\mathrm{B}a+\mathrm{C}$ may be positive, which it will of course necessarily be if b and c are imaginary; we must also have $a^2+\mathrm{B}a+\mathrm{C}-r$ positive, so that the form of φx is $m((x^2-a^2)+\mathrm{B}(x-a)-t)$, t being necessarily positive, but otherwise arbitrary, a form containing two arbitrary constants, one of which is subject to satisfy a certain condition of inequality; whereas when fx is of such a form as to *admit*, and $\varphi(x)$ is supposed to be so assumed as to *cause it to come to pass* that the quotients to $\frac{\varphi x}{fx}$ form a single sequence, then the three coefficients in φx remain exempt from all conditions of equality but are subject to two conditions of inequality. And so in general when the degree of fx is x and the number of sequences $2i+1$, it is to be inferred that the n coefficients of φx will be subject to satisfy $n-i-1$ conditions of inequality and i conditions of equality.

Art. (ξ.). The theory of the determination of the minimum interval between either limit determinable by this method and the nearest root, or between the two limits so determinable when φx is so assumed that $\frac{\varphi x}{fx}$ gives rise to a defined even number of sequences (which will include the theory of the case where all the roots of fx are imaginary), must be deferred to an opportunity more favourable for leisurely contemplation. As regards the application of the theory to the very interesting case of all the roots being imaginary, the principal point remaining to be cleared up is the determination of the least value that can be assigned to the greatest, and the greatest value that can be assigned to the least root of the algebraical product $\mathrm{X}_1.\mathrm{X}_2.\mathrm{X}_3\ldots\mathrm{X}_{2n}$, where $\mathrm{X}_1, \mathrm{X}_2, \ldots\mathrm{X}_{2n}$ are all of them real linear functions of x, subject to the condition that the cumulant $[\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3\ldots\mathrm{X}_{2n}]$ shall (to a numerical factor *près*) be equal to a given function of the degree $2n$ in x incapable of changing its sign, which condition implies, as a necessary consequence, that the coefficients of x in each of the terms $\mathrm{X}_1, \mathrm{X}_2, \ldots\mathrm{X}_{2n}$ must be affected with the same algebraical sign.

Art. (o.). It should be observed that in the application of the above method, the division of the series of quotients into distinct sequences governed by the signs of the coefficients of x is introduced for the purpose of drawing the limits closer to the roots, but is not necessary for the mere object of assigning limits.

Thus, for instance, if there be two sequences so that

$$[q_1 q_2 \dots q_i ; \quad q_{i+1} q_{i+2} \dots q_{i+i'}]$$

$$q_1^2 = \mu_1^2 \quad q_1^2 = \left(\mu_2 + \frac{1}{\mu_1}\right)^2 \quad q_2^2 = \left(\mu_3 + \frac{1}{\mu_2}\right)^2 \dots q_i^2 = \left(\frac{1}{\mu_{i-1}}\right)^2$$

and $$q_{i+1}^2 = \nu_1^2 \quad q_{i+2}^2 = \left(\nu_2 + \frac{1}{\nu_1}\right)^2 \dots q_{i+i'}^2 = \left(\frac{1}{\nu_{i'-1}}\right)^2$$

the greatest and least roots of x deduced from these equations will be superior and inferior limits respectively to the roots of fx; from which it is clear that if leaving all the other equations unaltered, except those which contain respectively q_i^2 and q_{i+1}^2, we write in place of these

$$q_i^2 = \left(\varrho + \frac{1}{\mu_{i-1}}\right)^2$$

$$q_{i+1}^2 = \left(\frac{1}{\varrho} + \nu_1\right)^2$$

the roots of the system of $i+i'$ equations thus modified will *à fortiori* be limits to the roots of fx, but then the quantities

$$\mu_1, \mu_2 + \frac{1}{\mu_1} \dots, \mu_{i-1} + \frac{1}{\mu_{i-2}}, \varrho + \frac{1}{\mu_{i-1}}, \nu_1 + \frac{1}{\varrho}, \nu_2 + \frac{1}{\nu_1}, \dots, \frac{1}{\nu_{i'-1}}$$

form the same single series as would correspond to the two sequences

$$q_1 q_2 \dots q_i q_{i+1} \dots q_{i+i'},$$

treated as a single sequence, and the same is obviously the case for any number of sequences*.

Art. (π.). If we consider a single sequence as $q_1 q_2 \dots q_n$, and write

$$q_1 = a_1(x - c_1) \quad q_2 = a_2(x - c_2) \dots q_n = a_n(x - c_n)$$

where $a_1, a_2, \dots, a_n$ are supposed to have all the same sign, and write

$$a_1^2(x-c_1)^2 = \mu_1^2 \quad a_2^2(x-c_2)^2 = \left(\mu_2 + \frac{1}{\mu_1}\right)^2 \dots a_n^2(x-c_n)^2 = \left(\frac{1}{\mu_{n-1}}\right)^2$$

* It follows from this, that if $q_1, q_2, \dots q_n$ be all linear functions of x, and if

$$Q = (q_1^2 - \mu_1^2)\left(q_2^2 - \left(\mu_2 + \frac{1}{\mu_1}\right)^2\right)\left(q_3^2 - \left(\mu_3 + \frac{1}{\mu_2}\right)^2\right) \dots \dots \left(q_n^2 - \frac{1}{\mu_{n-1}^2}\right),$$

no root of Q can lie between the extreme roots of the function K, used to denote the cumulant

$$[\sqrt{q_1^2}, \quad -\sqrt{q_2^2}, \quad \sqrt{q_3^2}, \dots \dots \dots, \pm\sqrt{q_n^2}],$$

the square roots being understood to be taken so as to make the sign of the coefficients of x all of them positive; and from a preceding article we know that either extreme root of Q can be made to coincide with a corresponding extreme root of K. Hence we have an *à priori* solution of the following question, viz. "To determine the $(n-1)$ positive quantities $\mu_1, \mu_2, \dots \mu_{n-1}$, so as to make the greatest root of Q a minimum and its least root a maximum;" for the greatest root of K will be the minimum greatest root of Q, and the least root of K the maximum least root of Q. Calling these respectively l and λ, the two systems of values of $\mu_1, \mu_2, \dots \mu_{n-1}$ required will be obtained by substituting respectively l and λ for x in the equations

$$\mu_1 = \sqrt{q_1^2} \quad \mu_2 = -\sqrt{q_2^2} - \frac{1}{\mu_1} \quad \mu_3 = +\sqrt{q_3^2} - \frac{1}{\mu_2} \dots \dots \dots \mu_{n-1} = \pm\sqrt{q_{n-1}^2} - \frac{1}{\mu_{n-2}}.$$

it seems not unlikely that the interval between the greatest and least of the roots of the above equations will be a minimum when the intervals between any pair is the same for each pair, *i.e.* when

$$\frac{\mu_1}{a_1}=\frac{\mu_2+\dfrac{1}{\mu_1}}{a_1}=\frac{\mu_3+\dfrac{1}{\mu_2}}{a_2}=\ldots=\frac{\dfrac{1}{\mu_{n-1}}}{a_n}.$$

If we assume these equations, and write $\mu_1=a_1\xi$, the equation for determining ξ, will be

$$[a_1\xi,\ a_2\xi,\ a_3\xi,\ \ldots,\ a_n\xi]=0.$$

If $n=2$ this equation becomes $a_1a_2\xi_2-1=0$.

If $n=3$, rejecting the factor ξ, it becomes

$$a_1a_2a_3\xi^2-(a_1+a_3)=0.$$

If $n=4$ it becomes

$$a_1.a_2.a_3.a_4\,\xi^4-(a_1.a_2+a_3.a_4+a_1.a_4)\xi^2+1=0.$$

If $n=5$, rejecting the factor ξ, it becomes

$$a_1.a_2.a_3.a_4.a_5\,\xi^4-(a_1.a_2.a_3+a_1.a_2.a_5+a_1.a_4.a_5+a_3.a_4.a_5)\xi^2+(a_1+a_3+a_5)=0,$$

and so in general the equation in ξ^2 being always of a degree measured by the integer nearest to and not exceeding $\frac{n}{2}$; and it is easy to be seen that for all values of n, the second coefficient divided by the first will be an inferior limit to ξ^2 (of course actually coinciding with it for the cases of $n=2$ and $n=3$). Hence we have the following valuable practical rule for finding a superior and inferior limit to the cumulant

$$[a_1(x-c_1),\ a_2(x-c_2),\ \ldots,\ a_n(x-c_n)],$$

where $a_1, a_2, \ldots a_n$ have the same sign, viz. if C be the greatest, and K be the least of the quantities $c_1, c_2, \ldots c_n$, $C+\Delta$ will be a superior, and $K-\Delta$ an inferior limit, Δ being taken equal to the positive value of

$$\sqrt{\frac{1}{a_1.a_2}+\frac{1}{a_2.a_3}+\frac{1}{a_3.a_4}+\cdots\frac{1}{a_{n-1}.a_n}};$$

and it may be noticed that C and K are the quantities which would themselves be the superior and inferior limits to the given cumulant if the series of terms $a_1, a_2, \ldots a_n$, instead of presenting only a sequence of continuations or permanencies, presented only a sequence of changes or variations of sign.

Section V.

On the Theory of Intercalations as applicable to two functions of the same degree, and on the formal properties of the Bezoutiant with reference to the method of Invariants.

Art. (56.). If fx and φx be any two given functions of x of the same degree m, we may form a system of m Bezoutics to f and φ (as shown in the first section), the coefficients of the powers of $x^{m-1}, x^{m-2}, \ldots x^1 x^0$ in which will compose a square matrix of m lines of m terms each, which will be symmetrical in respect to the diagonal

which passes through the first coefficient of the first Bezoutic and the last coefficient of the last Bezoutic; and we may construct a quadratic homogeneous function of m new variables, such that its determinantive matrix shall coincide with the Bezoutic square so formed. This quadratic form may be considered in the light of a generating function. All its coefficients will be formed of quantities obtained by taking any two coefficients in one of the given functions, and two corresponding coefficients in the other given function, multiplying them in cross order, and taking the difference: each coefficient of the generating function in question will consist of one or more such differences, and will thus be of two dimensions altogether, being linear in respect to the coefficients of f, and also linear in respect to the coefficients of φ. This generating function I term the *Bezoutiant*, and it may be denoted by the symbol $B(f, \varphi)$: the determinant of B is of course the resultant to f, φ, and the matrix to B is the Bezoutic square to f, φ. Now we have seen that the decrease in the number of continuations of sign in the series 1, $B_1(x)$, $B_2(x) \ldots B_m(x)$ (where $B_1(x)$, $B_2(x) \ldots B_m(x)$ are the (n) Bezoutics to f, φ), as x changes from a to b, measures the number of roots of fx retained in the effective scale of intercalations taken between the limits (a) and (b). If we take the entire scale between $+\infty$ and $-\infty$ the total number of effective intercalations will be the same, whether reckoned by the number of roots of f or of φ remaining; for these two numbers can never differ except by a unit, since no two of either can ever come together; but the number of each remaining in the effective scale will be $m-2i$ and $m-2i'$ respectively, i being the number of pairs of imaginary roots and pairs of unseparated real roots of f and i' being the similar number for φ; so that we must have $i=i'$.

Now obviously this number becomes measured by the number of continuations of sign in the *signaletic* series 1, (B_1), (B_2), ... (B_m), where in general (B_i) denotes the principal coefficient in $B_i(x)$.

But (B_1), (B_2), ... (B_m) are the successive ascending coaxal minor determinants about the axis of symmetry to the Bezoutic square; and accordingly the number of continuations just spoken of, measures the number of positive terms in the Bezoutiant when linearly transformed, so as to contain only positive and negative squares, or in other words, measures the *inertia* of the Bezoutiant, the constant integer which adheres to it under all its real linear transformations.

Art. (57.). This inertia is the same number as in the case of a homogeneous quadratic function of three variables, used to express a curve referred to trilinear coordinates, serves to determine whether such conic belongs to the impossible class or to the possible class of conics, being 3 or 0 in the former case, and 1 or 2 in the latter; or as in the case of a homogeneous quadratic function of four variables used to denote a surface referred to quadriplanar or tetrahedral coordinates, serves to determine whether such surface belongs to the impossible class or to the class consisting of the ellipsoid and the hyperboloid of two sheets (which are *descriptively* indistinguishable), or to the hyperboloid of one sheet, being 0 or 4 in the first case,

1 or 3 in the second, and 2 in the third. The most *symmetrical* (but least expeditious) method of finding the *inertia* of any quadratic form is that which corresponds to the method of orthogonal transformations, and is, in fact, the usual method employed in geometrical treatises on lines and surfaces of the second degree. If we apply this method to the Bezoutiant B considered as a homogeneous quadratic function of the (m) arbitrarily named variables $u_1, u_2, u_3, \ldots u_m$ in order to measure its inertia, that is to say, the number of effective interpositions between the two systems of roots, we must construct the determinant

$$\mathrm{D}(\lambda)=\begin{Bmatrix} \frac{d^2.\mathrm{B}}{du_1^2}+\lambda\,; & \frac{d^2\mathrm{B}}{du_1.du_2}\,; & \frac{d^2\mathrm{B}}{du_1.du_3}\,; & \cdots & \frac{d^2\mathrm{B}}{du_1.du_m} \\ \frac{d_2\mathrm{B}}{du_2.du_1}\,; & \frac{d^2\mathrm{B}}{du_2^2}+\lambda\,; & \frac{d^2\mathrm{B}}{du_2.du_3}\,; & \cdots & \frac{d^2\mathrm{B}}{du_2.du_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{d^2\mathrm{B}}{du_m.du_1}\,; & \frac{d^2\mathrm{B}_2}{du_m.du_2}\,; & \frac{d^2\mathrm{B}}{du_m.du_3}\,; & \cdots & \frac{d^2\mathrm{B}}{du_m^2}+\lambda \end{Bmatrix}$$

All the roots of $\mathrm{D}(\lambda)=0$, as is well known, are real; the inertia of B, being measured by the number of positive roots of $\mathrm{D}(-\lambda)$, will be equal to the number of continuations of sign in $\mathrm{D}(\lambda)$ expressed as a function of λ of the mth degree.

If in fx and φx we reverse the order of the coefficients, and fx and φx so transformed become $f_1(x)$ and $\varphi_1(x)$, it is obvious that the roots of f_1 and φ_1 being the reciprocals of the roots of f and φ respectively, the number of effective intercalations to f_1 and φ_1 must be the same as for f and φ. Accordingly we find that the form of the Bezoutiant to f and φ is the same as that of the Bezoutiant to f_1 and φ_1, the sole difference (one only of *names*) being that $\mathrm{B}(u_1, u_2, \ldots u_{m-1}, u_m)$ for the one becomes $\mathrm{B}(u_m, u_{m-1}, \ldots u_2, u_1)$ for the other. The equation $\mathrm{D}(\lambda)$, which determines the *inertia* of B, remains precisely the same as it ought to do for either of the two systems f and φ or f_1 and φ_1.

Art. (58.). The theory in the preceding articles of this section may be made to embrace the case involved in Sturm's theorem; for if

$$fx=a_0.x^n+a_1.x^{n-1}+\ldots+a_{m-1}.x^{m-1}+a_m.x^m$$

$$f'x=ma_0.x^{n-1}+(n-1)a_1.x^{n-2}+\ldots+a_{m-1},$$

and

$$f_1x=mfx-f'x$$

$$=a_1.x^{n-1}+2a_2.x^{n-2}+\ldots+m.a_m$$

the Bezoutian secondaries, or which is the same thing, the simplified Sturmian residues to fx and $f'x$, will evidently be the same as those to f_1x and $f'x$. Accordingly, if we form the signaletic series

$$fx, f'x, \mathrm{B}_1, \mathrm{B}_2 \ldots \mathrm{B}_{m-1},$$

where $B_1, B_2 \ldots B_{m-1}$ are the Bezoutian secondaries to f_1x and $f'x$, the number of *variations* of sign between consecutive terms in this series, when x is made $+\infty$, will measure the number of pairs of imaginary roots in fx; and fx and $f'x$ forming always a continuation, and the coefficient of $f'(x)$ being supposed positive, we see that the terms of the rhizoristic series will be 1, (B_1), $(B_2) \ldots (B_{m-1})$ consisting of positive unity, and the successive ascending coaxal determinants of the Bezoutian matrix to f' and f_1x. Hence then the form of the Bezoutiant to $f'x$ and f_1x will serve to determine the number of pairs of imaginary, and consequently also the number of real roots to fx. It should be remarked that the form of the Bezoutiant to $f'x$ and f_1x, considered as a quadratic function of $u_1, u_2 \ldots u_{m-1}$ and of the coefficients in $f(x)$, will remain unaltered when for fx we write f_1x, for this will change the signs throughout of fx and f_1x, and consequently the coefficients in the Bezoutiant, which contain in every term one coefficient from $f'x$, and one from f_1x, will remain unaltered in sign.

Art. (59.). It appears then from the preceding article, that for every function of x of the degree m, there exists a homogeneous quadratic function of $(m-1)$ variables, the *inertia* of which augmented by unity will represent the number of real roots in the given function. Now this *inertia* itself may be measured by the number of positive roots of a certain equation in λ formed from the quadratic function (in fact the well-known equation for the secular inequalities of the planets), all whose roots will be real. Hence then we are led to the following remarkable statement. "*An algebraical equation of any degree being given, an equation whose degree is one unit lower may be formed, all the roots of which shall be real, and of which the number of positive roots shall be one less than the total number of real roots of the given equation.*"

Let us suppose fx written in its most general form, the first and last as well as all the intermediate coefficients being anything whatever: by reversing the order of the coefficients $f'x$ will become f_1x and f_1x will become $f'x$; the Bezoutiant to f_1x and $f'x$ (which we may term the Bezoutoid to fx) will remain unaltered except in sign, and the equation of the $(m-1)$th degree in λ formed from the Bezoutoid remain unchanged, consequently the equation in λ enables us to substitute, for the purpose of calculating the total number of real roots in $f(x)$ in lieu of STURM's auxiliary functions to $f(x)$, another set of functions which remain unaltered when the order of the coefficients is completely reversed, *i. e.* in effect, when we consider the number of real roots of $f\left(\frac{1}{x}\right)$ in lieu of those of $f(x)$. And of course more generally the equation of the mth degree in λ formed from the Bezoutiant to any two functions fx and ϕx of the mth degree each in x, supplies a set of functions for determining the total number of effective intercalations between the roots of $f(x)$ and $\phi(x)$, which do not alter when we consider in lieu of these, the roots of $f\left(\frac{1}{x}\right)$ and $\phi\left(\frac{1}{x}\right)$. This substitution of functions symmetrically formed in respect to the two ends of an equation for the purpose of assigning the total number of real roots in lieu of the unsymmetrical ones furnished

by the ordinary method of M. STURM, had been long felt by me to be a desideratum, and as an object the accomplishment of which was indispensable to the ulterior development of the theory, and it is certain that I did not in anticipation exaggerate the importance of the result to be attained.

Art. (60.). It may happen that the Bezoutiant to f and φ (each of the mth degree) may become a quadratic function of less than m independent variables, or the Bezoutoid to f (a function in x of the mth degree) of less than $(m-1)$ independent variables. This will take place whenever f and φ have roots in common, or whenever F has equal roots. The number of independent relations of equality between the roots of f and φ, and the amount of multiplicity, however distributed, among the roots of F, will be indicated by the number of *orders* thus disappearing out of the general form of the Bezoutiant and Bezoutoid in the respective cases*. In what particular mode the form of each would be affected according to the manner of the distribution of the equalities and the multiplicity requires a specific discussion, which I must reserve for some future occasion.

Art. (61.). I shall devote the remainder of this memoir to a consideration of the properties and affinities of Bezoutiants or Bezoutoids, regarded from the point of view of the Calculus of Invariants. For this purpose it will be more convenient hereafter to convert all the functions which we are concerned with into homogeneous forms, and I shall accordingly for the future use f and φ to denote functions each of x and y, which I shall write under the form

$$f=a_0.x^m+ma_1.x^{m-1}.y+m.\frac{m-1}{2}a_2.x_{m-2}.y^2+\ldots+a_m.x^m$$

$$\varphi=b_0.x^m+b_1.x^{m-1}.y+m.\frac{m-1}{2}b_2.x_{m-2}.y^2+\ldots+b_m.x^m.$$

In what follows a knowledge of the general principles of the Method of Invariants is presupposed, but a perusal of my two papers on the Calculus of Forms in the Cambridge and Dublin Mathematical Journal, February and May 1852, will furnish nearly all the information that is strictly necessary for the present purpose. The first point to be established is, that B, the Bezoutian of fx and φx, is a Covariant to the system f, φ; the variables in B being in compound relation of cogredience with the combinations of powers of x and y,

$$x^{m-1};\ x^{m-2}.y;\ x^{m-3}\ldots y^{m-1}.$$

That is to say, I propose to show that if f, g, h, k be any four quantities, taken for greater simplicity subject to the relation $fk-gh=1$, and if on substituting $fx+gy$ for x and $hx+ky$ for y, $f(x, y)$ becomes

$$A_0.x^m+mA_1.x^{m-1}.y+m.\frac{m-1}{2}A_2.x^{m-2}.y^2+A_m.y^m, \text{ say } G(x, y),$$

* I have elsewhere defined how this word order, as here employed, is to be understood. If F, a homogeneous function of $x_1, x_2, \ldots x_m$ can be expressed as a function of $u_1, u_2, \ldots u_{m-i}$ (all linear functions of $x_1, x_2, \ldots x_m$), F is said to be a function of $n-i$ orders, or to have lost i of the orders belonging to the complete form.

and $\varphi(x, y)$ becomes

$$B_0.x^m+B_1.x^{m-1}.y+m.\frac{m-1}{2}B_2x^{m-2}.y^2+B_m.y^m, \text{ say } T(x, y),$$

and if $B'(u'_1, u'_2 \ldots u'_m)$ be the Bezoutiant to G and T; $B(u_1, u_2 \ldots u_m)$ being that to f and φ, then, on making $u_1, u_2 \ldots u_m$, the same *linear* functions of $u'_1, u'_2 \ldots u_m$ as $(fx+gy)^m$; $(fx+gy)^{m-1}(hx+ky)$; .....$(fx+gy)(hx+ky)^{m-1}$; $(hx+ky)^{m-1}$ are respectively of

$$x^m, x^{m-1}.y \ldots x.y^{m-1}; y^m,$$

B will become identical with B'. I was led to suspect the high probability of the truth of this proposition concerning the invariance of the Bezoutiant from the following considerations: 1st. That for the particular case where f and φ are the differential derivatives in respect to x and y respectively of the same function $F(x, y)$, the Bezoutiant of f and φ, which then becomes the Bezoutoid of F, determines the number of real factors in F, which obviously remains the same for all linear transformations of F. 2ndly. That taking f and φ in their most general form, the invariant to their Bezoutiant, *i. e.* the determinant of their Bezoutiant is an invariant of f and φ, being in fact the resultant of these two functions; now as every concomitant (an invariantive form of the most general kind) to a concomitant is itself a concomitant to the primitive, so it appeared to me, and is I believe true (although awaiting strict proof), that any form satisfying certain necessary and tolerably obvious conditions of homogeneity and isobarism, a concomitant to which is also a concomitant to a given form, will be itself a concomitant to such form; this principle, if admitted, would be of course at once conclusive as to the Bezoutiant being an invariantive concomitant to the functions from which it is derived.

Art. (61*.). Since the publication of the two papers above referred to on the Calculus of Forms, I have made the important observation that every species of concomitant, however complex, to a given system of functions, may be treated as a simple invariant of a system including the given system together with an appropriate superadded system of absolute functions; thus an ordinary covariant involving only one system of variables, as $u, v, w \ldots$ cogredient with $x, y, z \ldots$ the variables of a system S, is in fact an invariant of the system S combined with the system $ux-vy, vz-wy, wx-uz$, &c., $u, v, w \ldots$ being treated as constants; so again a simple contravariant of S is an invariant of S combined with the equation $ux+vy+wz+$&c.; so again, to meet the case before us, a covariant to the binary system f and φ expressed as a function of $u_1, u_2 \ldots u_m$, where $u_1, u_2 \ldots u_m$ are cogredient with $x^{m-1}, x^{m-2}.y \ldots y^{m-1}$, may be regarded as an invariant of the ternary system f, φ, Ω, where

$$\Omega=u_1.y^{m-1}-mu_2.y^{m-2}x+m.\frac{m-1}{2}u_3.y^{m-3}x^2 \ldots +(-)^{m-1}.u_{m-1}.x^{m-1},$$

($u_1, u_2, \ldots u_{m-1}$ being here to be treated as constants), and accordingly the differential equations which serve to define in the most general and absolute manner such cova-

riant of f, φ, or invariant to f, φ, Ω, say I, will take the form

$$\left\{\begin{array}{l}\left\{\left(a_0.\frac{d}{da_1}+b_0.\frac{d}{db_1}\right)+2\left(a_1.\frac{d}{da_2}+b_1.\frac{d}{db_2}\right)+3\left(a_2.\frac{d}{da_3}+b_2.\frac{d}{db_3}\right)+\ldots+m\left(a^{m-1}.\frac{d}{da_m}+b_{m-1}.\frac{d}{db_m}\right)\right.\\ \left.\quad-\left(u_1\frac{d}{du_2}+2u_2.\frac{d}{du_3}+3u_3.\frac{d}{du_4}+\ldots+(m-1)u_{m-1}.\frac{d}{du_m}\right)\right\}\end{array}\right\}\mathrm{I}=0$$

$$\left\{\begin{array}{l}\left\{a_m.\frac{d}{da_{m-1}}+b_m.\frac{d}{db_{m-1}}\right\}+2\left\{a_{m-1}.\frac{d}{da_{m-2}}+b_{m-1}.\frac{d}{db_{m-2}}\right\}\\ \quad+3\left\{a_{m-2}.\frac{d}{da_{m-3}}+b_{m-2}.\frac{d}{db_{m-3}}\right\}+\ldots+m\left(a_1.\frac{d}{da_0}+b_1.\frac{d}{db_0}\right)\\ \quad-\left(u_{m-1}.\frac{d}{du_{m-2}}+2u_{m-2}.\frac{d}{du_{m-3}}+3u_{m-3}.\frac{d}{du_{m-4}}\ldots+(m-1)a_2.\frac{d}{du_1}\right)\end{array}\right\}\mathrm{I}=0.$$

These equations may be proved to be satisfied when I is taken $=$B, the Bezoutiant to f, φ, and thus B may be proved to be a covariant to f, φ, but the demonstration is long and tedious. An admirable suggestion, well worthy of its keen-witted author, for which I am indebted to Mr. Cayley, will enable us to prove the invariantive character of B by a much more expeditious method.

Art. (62.). For greater simplicity begin with considering functions of a single variable x; and in order to fix the ideas, suppose (m) to be taken 5, and write

$$fx=ax^5+bx^4+cx^3+dx^2+ex+l$$
$$\varphi x=\alpha x^5+\beta x^4+\gamma x^3+\delta x^2+\varepsilon x+\lambda,$$

and let $\mathfrak{D}=\frac{fx\varphi x'-fx'.\varphi x}{x-x'}$; this is of course an integral function of x and x', since the numerator vanishes when $x=x'$; and we have by performing the actual operations,

$$\begin{aligned}\mathfrak{D}=&\left\{\begin{array}{l}(a\beta-b\alpha)x^4.x'^4+(a\gamma+c\alpha)x^3.x'^3(x+x')+(a\delta-d\alpha)x^2x'^2(x^2+xx'+x'^2)+(a\varepsilon-e\alpha)\\ \quad xx'(x^3+x^2x'+xx'^2+x'^3)+(a\lambda-l\alpha)(x^4+x^3x'+x^2x'^2+xx'^3+x'^4)\end{array}\right\}\\ &+\left\{\begin{array}{l}(b\gamma-c\beta)x^3x'^3+(b\delta-d\beta)x^2x'^2(x+x')+(b\varepsilon-e\beta)xx'(x^2+xx'+x'^2)\\ \quad+(b\lambda-l\beta)(x^3+x^2x'+xx'^2+x'^3)\end{array}\right\}\\ &+\left\{(c\delta-d\gamma)x^2x'^2+(c\varepsilon-e\gamma)xx'(x+x')+(c\lambda-l\gamma).(x^2+xx'+x'^2)\right)\\ &+\left((d\varepsilon-e\delta)xx'+(d\lambda-l\delta)(x+x')\right)\\ &+(e\lambda-l\varepsilon);\end{aligned}$$

and if we arrange $\mathfrak{D}$ under the form

$$\begin{array}{llllll}\mathrm{A}_{4,4}x^4.x'^4+\mathrm{A}_{4,3}x^4.x'^3+\mathrm{A}_{4,2}x^4x'^2+\mathrm{A}_{4,1}x^4.x'+\mathrm{A}_{4,0}.x^4\\ +\mathrm{A}_{3,4}x^3x'^4+\mathrm{A}_{3,3}x^3x'^3+\mathrm{A}_{3,2}x^3x'^2+\mathrm{A}_{3,1}x^3x'+\mathrm{A}_{3,0}x^3\\ +\mathrm{A}_{2,4}x^2.x'^4+\mathrm{A}_{2,3}x^2.x'^3+\mathrm{A}_{2,2}x^2x'^2+\mathrm{A}_{2,1}x^2x'+\mathrm{A}_{2,0}x^2\\ +\mathrm{A}_{1,4}xx'^4+\mathrm{A}_{1,3}xx'^3+\mathrm{A}_{1,2}xx'^2+\mathrm{A}_{1,0}xx'+\mathrm{A}_{1,0}x\\ +\mathrm{A}_{0,4}x'^4+\mathrm{A}_{0,3}x'^3+\mathrm{A}_{0,2}x'^2+\mathrm{A}_{0,1}x'+\mathrm{A}_{0,0};\end{array}$$

it will readily be perceived that the matrix formed by the twenty-five coefficients, viz.—

$$\begin{matrix} A_{4,4} & A_{4,3} & A_{4,2} & A_{4,1} & A_{4,0} \\ A_{3,4} & A_{3,3} & A_{3,2} & A_{3,1} & A_{3,0} \\ A_{2,4} & A_{2,3} & A_{2,2} & A_{2,1} & A_{2,0} \\ A_{1,4} & A_{1,3} & A_{1,2} & A_{1,1} & A_{1,0} \\ A_{0,4} & A_{0,3} & A_{0,2} & A_{0,1} & A_{0,0}, \end{matrix}$$

will be symmetrical about its dexter diagonal (that one, namely, which passes through $A_{4,4}$ and $A_{0,0}$), and will be identical with the Bezoutian square corresponding to the system f, φ; in fact, using the notation previously employed in the first section, it becomes

$$(\alpha.) \qquad \begin{matrix} (0,1) & (0,2) & (0,3) & (0,4) & (0,5) \\ (0,2) & \left\{\begin{matrix}(0,3)\\+\\(1,2)\end{matrix}\right\} & \left\{\begin{matrix}(0,4)\\+\\(1,3)\end{matrix}\right\} & \left\{\begin{matrix}(0,5)\\+\\(1,4)\end{matrix}\right\} & (1,5) \\ (0,3) & \left\{\begin{matrix}(0,4)\\+\\(1,3)\end{matrix}\right\} & \left\{\begin{matrix}(0,5)\\+\\(1,4)\\+\\(2,3)\end{matrix}\right\} & \left\{\begin{matrix}(1,5)\\+\\(2,3)\end{matrix}\right\} & (2,5) \\ (0,4) & \left\{\begin{matrix}(0,5)\\+\\(1,4)\end{matrix}\right\} & \left\{\begin{matrix}(1,5)\\+\\(2,3)\end{matrix}\right\} & \left\{\begin{matrix}2,5\\+\\3,4\end{matrix}\right\} & (3,5) \\ (0,5) & (1,5) & (2,5) & (3,5) & (4,5), \end{matrix}$$

(r, s) being used in general to denote the difference between the cross products of the coefficients of x^{s-r} and x^{s-s} in f and φ. Restoring now to m its general value, and taking f and φ homogeneous functions of x and y, and making

$$\Im = \frac{f(x,y)\varphi(x',y') - f(x',y')\varphi(x,y)}{xy' - x'y},$$

we see without difficulty that

$$\Im = \Sigma A_{r,s}\{x^r.y^{m-1-r}x^s.y^{m-1-s}\};$$

where $A_{r,s}$ is the term in the rth line and sth column of the Bezoutiant matrix to f and φ. This is the identification, the idea of which, as before observed, is due to Mr. Cayley.

Art. (63.). If, now, we consider the system of functions

$$f(x,y) = a_0.x^m + ma_1.x^{m-1}.y + \ldots\ldots\ldots\ldots + a_m.y^m$$
$$\varphi(x,y) = b_0.x^m + mb_1.x^{m-1}.y + \ldots\ldots\ldots\ldots + b_m.y^m$$
$$\Omega(x,y) = u_{m-1}.y^{m-1} - (m-1)u_{m-2}.y^{m-2} \pm \ldots\ldots\ldots + (-)^{m-1}.u_1.x^{m-1},$$

evidently $f(x, y)\varphi(x', y')-f(x', y')\varphi(x, y)$ is a covariant with f and φ, and therefore (which is a mere truism) with the entire system f, φ, Ω. So also is $xy'-x'y$, and therefore $\mathfrak{S}$, the quotient of these two, is a covariant to the system. Hence, therefore, by virtue of a general theorem given in my Calculus of Forms,

$$\Omega\left(\frac{d}{dy}, -\frac{d}{dx}\right)\mathfrak{S}$$

is a covariant to the system; and again, therefore,

$$\Omega\left(\frac{d}{dy'}, -\frac{d}{dx'}\right).\Omega\left(\frac{d}{dy}, -\frac{d}{dx}\right)\mathfrak{S}$$

is a covariant thereto. Now $\mathfrak{S}$ is of $(m-1)$ dimensions in x, y and also of the same in x', y'. Consequently this latter form will contain only the quantities $u_1, u_2, \ldots u_{m-1}$, and the coefficients of f and φ, so that the powers of x, y; x', y' will not appear in it.

Now $$\mathfrak{S}=\Sigma^{0}_{m-1}\Sigma^{0}_{m-1}A_{r,s}\{x^{r}.y^{m-1-r}.x'^{s}.y'^{m-1-s}\}$$

$$(-)^{m-1}\Omega\left(\frac{d}{dy}, -\frac{d}{dx}\right)=u_{m-1}\left(\frac{d}{dx}\right)^{m-1}+(m-1)u_{m-2}\left(\frac{d}{dx}\right)^{m-2}\frac{d}{dy}+\ldots+u_1\left(\frac{d}{dy}\right)^{m-1}$$

$$(-)^{m-1}\Omega\left(\frac{d}{dy'}, -\frac{d}{dx'}\right)=u_{m-1}\left(\frac{d}{dx'}\right)^{m-1}+(m-1)u_{m-2}\left(\frac{d}{dx'}\right)^{m-2}\frac{d}{dy'}+\ldots+u_1\left(\frac{d}{dy'}\right)^{m-1},$$

$$\therefore \frac{1}{1.2.3\ldots(m-1)^2}.\Omega\left(\frac{d}{dy'}, -\frac{d}{dx'}\right).\Omega\left(\frac{d}{dy}, -\frac{d}{dx}\right)\mathfrak{S}$$

$$=\Sigma^{0}_{m-1}(A_{r,r}.u_r^2)+2\Sigma^{0}_{m-1}\Sigma^{0}_{m-1}(A_{r,s}.u_r.u_s),$$

r and s being excluded in the latter sum from being made equal; but this latter expression is the Bezoutiant to f, φ. Hence the Bezoutiant of f, φ is an invariant to f, φ, Ω, *i. e.* a covariant to the system f, φ, as was to be proved. The mode of obtaining the covariant $\mathfrak{S}$, used in this and the preceding article, is very remarkable. I believe that the true *suggestive* view of the process for finding it, is to consider

$$f(x, y).\varphi(x', y')-f(x', y').\varphi(x, y)$$

as a concomitant capable of being expressed under the form of a function of $\mathfrak{S}$ and ω, ω standing for the universal covariant $xy'-x'y$; $\mathfrak{S}$ is then to be considered, not properly as a quotient, but rather as an invariant of the form $\mathfrak{S}.\omega$, a function of ω of the first degree, where $\mathfrak{S}$ is treated as constant.

Art. (64.). B is not an ordinary covariant of f and φ, it belongs to that special and most important family of invariants to a system to which I have given the name of Combinants*, viz. Invariants, which, besides the ordinary character of invariance, when linear substitutions are impressed upon the variables, possess the same character of invariance when linear substitutions are impressed upon the functions themselves containing the variables; combinants being, as it were, invariants to a system of

* For some remarks on the Classification of Combinants, see Cambridge and Dublin Mathematical Journal, November, 1853.

functions in their corporate combined capacity *qud* system. That the Bezoutiant possesses this property is evident; for if instead of f and φ we write $kf+i\varphi$ and $k'f+i'\varphi$, any such quantity as $a_r.b_s-a_s.b_r$ (a_r, b_r being coefficients in f, and a_s, b_s the corresponding ones in φ) becomes

$$(ka_r+ib_r)(k'a_s+i'b_s)-(ka_s+ib_s)(k'a_r+i'b_r),\ i.\ e.\ (ki'-k'i)(a_r.b_s-a_s.b_r),$$

so that B, the Bezoutiant, becomes increased in the ratio of $(ki'-k'i)^m$, *i. e.* remains always unaltered in point of form and absolutely immutable, provided that $ki'-k'i$ be taken, as we may always suppose to be the case, equal to 1.

We derive immediately from this observation, the somewhat remarkable geometrical proposition, that the intersections with the axis of x made by any two curves of the family of curves $u=\lambda f(x)+\mu\varphi(x)$, ($f$ and φ being functions of x of the same degree) give rise to a constant number of effective intercalations, whatever values be given to λ or μ for the two curves so selected.

Art. (65.). $B(u_1, u_2, \ldots u_m)$ being a covariant of the system f and φ, and $u_1, u_2, \ldots u_m$ cogredient with $x^{m-1}, x^{m-2}.y, \ldots y^{m-1}$, it follows from a general principle in the theory of invariants, that on making $u_1, u_2, \ldots u_m$ respectively equal to the quantities with which they are cogredient, B will become an ordinary covariant to f and φ. By this transformation B becomes a function of x and y of the degree $2(m-1)$ in x and y conjointly, and linear in respect to the coefficients of f, and also in respect to those of φ. The only covariant capable of answering this description is what I am in the habit of calling the Jacobian (after the name of the late but ever-illustrious Jacobi), a term capable of application to any number of homogeneous functions of as many variables. In the case before us, where we have two functions of two variables, the Jacobian

$$J(f,\varphi)=\begin{Bmatrix}\frac{df}{dx}; & \frac{d\varphi}{dx}\\ \frac{df}{dy}; & \frac{d\varphi}{dy}\end{Bmatrix}=\frac{df}{dx}.\frac{d\varphi}{dy}-\frac{df}{dy}.\frac{d\varphi}{dx}.$$

We have then the interesting proposition*, that the Bezoutiant to two functions, when the variables in the former are replaced by the combinations of the variables in the latter, with which they are cogredient, becomes the Jacobian†. So in the case of a single function F of the degree m, the Bezoutiant, *i. e.* the Bezoutoid to $\frac{dF}{dx}, \frac{dF}{dy}$, on making the $(m-1)$ variables which it contains identical with $x^{m-2}; x^{m-3}.y; \ldots y^{m-2}$ respectively, becomes identical with the Jacobian to $\frac{dF}{dx}, \frac{dF}{dy}$, *i. e.* the Hessian of F, viz.

$$\begin{Bmatrix}\frac{d^2F}{dx^2}; & \frac{d^2F}{dxdy}\\ \frac{d^2F}{dxdy}; & \frac{d^2F}{dy^2}\end{Bmatrix}.$$

* I have subsequently found that this proposition is contained under another mode of statement, at the end of Section 2 of the Memoir of Jacobi, "De Eliminatione," above referred to.

† For a strict proof of this proposition see Supplement to Third Section of this memoir.

As an example of this property of the Bezoutiant, suppose

$$f=ax^3+bx^2y+cxy^2+dy^3$$
$$\varphi=\alpha x^3+\beta x^2y+\gamma xy^2+\delta y^3.$$

The Bezoutiant matrix becomes

$$\begin{array}{lll} a\beta-b\alpha\,; & a\gamma-c\alpha\,; & a\delta-d\alpha \\ & a\delta-d\alpha & \\ a\gamma-c\alpha\,; & + & ;\quad b\gamma-c\beta \\ & b\gamma-c\beta & \\ a\delta-d\alpha\,; & b\gamma-c\beta\,; & c\delta-d\gamma. \end{array}$$

The Bezoutiant accordingly will be the quadratic function

$$(a\beta-b\alpha)u_1^2+(a\delta-d\alpha+b\gamma-c\beta)u_1u_2+c\delta-d\gamma u_2^2$$
$$+2(a\gamma-c\alpha)u_1.u_3+2(a\delta-d\alpha)u_2.u_1+2(b\gamma-c\beta)u_2.u_3,$$

which on making

$$u_1=x^2 \quad u_2=xy \quad u_3=y^2,$$

becomes

$$\mathrm{L}x^4+\mathrm{M}x^3y+\mathrm{N}x^2y^2+\mathrm{P}xy^3+\mathrm{Q}y^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta.)$$

where L, M, N, P, Q respectively will be the sum of the terms lying in the successive bands drawn parallel to the sinister diagonal of the Bezoutiant matrix, *i. e.*

$$\mathrm{L}=a\beta-b\alpha$$
$$\mathrm{M}=2(a\gamma-c\alpha)$$
$$\mathrm{N}=3(a\delta-d\alpha)+(b\gamma-c\beta)$$
$$\mathrm{P}=2(b\gamma-c\beta)$$
$$\mathrm{Q}=c\delta-d\gamma.$$

The biquadratic function in x and y ($\beta.$) above written will be found on computation to be identical in point of form with the Jacobian to f, φ, viz.

$$(3ax^2+2bxy+cy^2)(\beta x^2+2\gamma xy+3\delta y^2)-(3\alpha x^2+2\beta xy+\gamma y^2)(bx^2+2cxy+dy^2),$$

this latter being in fact

$$3\mathrm{L}x^4+3\mathrm{M}x^3y+3\mathrm{N}x^2y^2+3\mathrm{P}xy^3+3\mathrm{Q}y^4.$$

The remark is not without some interest, that in fact the Bezoutiant, which is capable (as has been shown already) of being mechanically constructed, gives the best and readiest means of calculating the Jacobian; for in summing the sinister bands transverse to the axis of symmetry the only numerical operation to be performed is that of addition of positive integers, whereas the direct method involves the necessity of numerical subtractions as well as additions, inasmuch as the same terms will be repeated with different signs. Thus if

$$f=ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+ly^5$$
$$\varphi=\alpha x^5+\beta x^4y+\gamma x^3y^2+\delta x^2y^3+\varepsilon xy^4+\lambda y^5,$$

using (r, s) in the ordinary sense that has been considered throughout, we obtain by

taking the sum of the sinister bands in (α.)* for the value of B when we write x^4, x^3y, x^2y^2, xy^3, y^4 in place of u_1, u_2, u_3, u_4, u_5,

$$(0, 1)x^5+2(0, 2)x^2y+\big(3(0, 3)+(1, 2)\big)x^3y^2+\big(4(0, 4)+2(1, 3)\big)x^2y^3$$
$$+\big(5(0, 5)+3(1, 4)+(2, 3)\big)xy^4+\big(4(1, 5)+2(2, 4)\big)x^2y^5+\big(3(2, 5)+(3, 4)\big)x^3y^6$$
$$+2(3, 5)xy^7+(4, 5)y^8.$$

The direct process requires the calculation of

$$(5ax^4+4bx^3y+3cx^2y^2+2dxy^3+ey^4)(\beta x^4+2\gamma x^3y+3\delta x^2y^2+4\varepsilon xy^3+5\lambda y^4)$$
$$-(5\alpha x^4+4\beta x^3y+3\gamma x^2y^2+2\delta xy^3+\varepsilon y^4)(bx^4+2cx^3y+3dx^2y^2+4exy^3+5ly^4),$$

each coefficient of which will contain the numerical factor 5; so that to reduce the Jacobian to its simplest form each coefficient will necessitate the employment of additions, subtractions, and a division, instead of additions merely, as when the Bezoutic square is employed. For instance, to find the coefficient of $x^4.y$ from the above expression (α.), we have to calculate

$$\tfrac{1}{5}\big(25(0, 5)+16(1, 4)+9(2, 3)+4(3, 2)+(4, 1)\big),$$

$$i.e.\ \tfrac{1}{5}\big(25(0, 5)+(16-1)(1, 4)+(9-4)(2, 3)\big),$$

which is $5(0, 5)+3(1, 4)+(2, 3)$, agreeing with what has been found above for the value of such coefficient, by a simple process of counting. The same remark will, of course, also apply to the computation of the Hessian of F by means of its Bezoutoid.

(Art. 66.). This relation between the Bezoutiant and the Jacobian led me to inquire whether, as would at first sight appear probable, the Bezoutiant were the only lineo-linear quadratic function of (m) variables covariantive to f and φ (the word lineo-linear being used to denote the form of coefficients, such as those in the Bezoutiant, linear in respect of the coefficients in f and the coefficients of φ). If so, then there would have existed a method of performing the inverse process of recovering the Bezoutiant from the Jacobian, almost as simple as that of deriving the Jacobian from the Bezoutiant. On investigating the matter, however, I found that such is by no means the case†, but that there exists a whole family of independent lineo-

* *Vide* art. 62.

† This might have been concluded immediately from the following observation. Let J, the Jacobian of f and φ, be expressed under the form

$$A_0x^{2m-2}+(2m-2)A_1.x^{2m-1}.y+(2m-2)\frac{2m-3}{2}A_2x^{2m-2}.y^2+\ldots+A_{2m-2}.y^{2m-2},$$

then we know from the Calculus of Forms, that, D being taken to represent the persymmetrical Determinant

$$\begin{array}{llll} A_0; & A_1; & A_2; & \ldots\ldots; A_{m-1} \\ A_1; & A_2; & A_3; & \ldots\ldots; A_m \\ A_2; & A_3; & A_4; & \ldots\ldots; A_{m+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_{m-1}; & A_m; & A_{m+1}; & \ldots\ldots; A_{2m-2}, \end{array}$$

linear quadratic covariants of m variables to every two homogeneous functions of x and y of the mth degree. I have, moreover, I believe, succeeded in determining the number of such lineo-linear quadratic forms for any value of (m), of which all the rest, in whatever manner obtained, may be expressed as linear functions, the coefficients of the linear relations moreover being abstract numbers; in other words, I have succeeded in forming the fundamental or constituent scale of lineo-linear quadratic forms of m variables covariantive to f and ϕ; a result of too great interest, as exhibiting the affinities of the Bezoutiant to its cognate forms, to be altogether passed over in silence. Supposing the number of linearly independent forms of the kind to be ν, then speaking *à priori* any of the forms taken at random might seem to be equally eligible to form one of the ν included in the fundamental scale, combined with any $(\nu-1)$ others independent *inter se*, and of which the selected one is also independent. In fact, however, this is not so; for it will always be more satisfactory to contemplate the fundamental scale of forms as generated successively or simultaneously by a uniform process; and in the case before us, the process which I have hit upon, and which I believe is the simplest that can be employed for generating the fundamental scale, will be found not to include directly the Bezoutiant among the number. There will thus arise two subjects of inquiry; 1st, the mode of forming the fundamental scale, and proving its fundamental

$D=0$ is the condition to be satisfied in order that J may be representable under the form of the sum of the squares of $(m-1)$ linear functions of x and y, and D itself is an invariant to J, and consequently an invariant and (as is obvious from its form) a combinantive invariant to f and ϕ. Moreover, which is more immediately to the point, we know that the quadratic form Q

$$\left(A_0u_1^2+2A_1(u_1.(m-1)u_2)+A_2\left\{((m-1)u_2)^2+2u_1.\left(\frac{(m-1)(m-2)}{2}\right)u_3\right\}+\&c.+A_{2m-2}u_m^2\right)$$

will be an invariant to f, ϕ and Ω (this last quantity Ω being defined as in p. 524), and a combinantive covariant to f and ϕ in the same sense precisely as the Bezoutiant is a covariant to the same, and like the Bezoutiant is lineo-linear in respect of the coefficients of f and ϕ. If we operate with the symbol E, where E represents

$$\frac{d}{dA_0}v_1^2+2\frac{d}{dA_1}.v_1v_2+\frac{d}{dA_2}(v_2^2+2v_1.v_3)+\&c.+\frac{d}{dA_{2m-2}}.v_m^2,$$

upon K any invariant of f and ϕ, we shall obtain $E.K$, a quadratic function of $v_1v_2\ldots v_m$, which by the rules of the Calculus of Forms we know will be a contravariant to f and ϕ, and the matrix corresponding to which must evidently be persymmetrical. It is an interesting subject of inquiry, which I reserve for some future occasion, to determine the Co-bezoutiant, the Discriminant of which must be employed for K, so that when this discriminant is operated upon by E, the matrix corresponding to $E.K$ may become identical (term for term) with the matrix which is the inverse to the Bezoutiant matrix, which inverse, as JACOBI has so simply and beautifully demonstrated, possesses this persymmetrical character. *Vide* the "De Eliminatione," section 5. The investigation of the arithmetical connexion between the Q of this note and the fundamental Co-bezoutiants must be also similarly reserved. I believe it to be generally true, and have verified the fact for the case of two cubic functions, that $E.Q$ gives a quadratic form such that the corresponding matrix is the inverse to the matrix of Q. The calculations necessary for extending the verification of this remarkable proposition for functions of x, y exceeding the third degree (notwithstanding that they are much abbreviated by the application of the rules of the calculus) still remain excessively laborious. The abbreviation alluded to consists in confining the verification in question to the comparison of either one of the two unreiterated terms at opposite corners of the matrix to $E.Q$ with the corresponding term in the inverse matrix of Q; if these coincide, it is easy to prove that every other pair of corresponding terms in the two matrices must also coincide respectively with one another.

character; 2ndly, determining the numerical relations which connect that very important form, perhaps of all of its kind, the most important with the forms comprised in the fundamental or constituent scale. These questions I propose to consider more fully at a future period. For the present I shall content myself with giving a method of forming the constituent scale (without, however, seeking the proof of all the forms *extra* to such assumed scale being linear functions of these comprised within it), and with determining the numerical relations between the forms in this scale and the Bezoutiant for a limited number of values of m. All the forms which we are seeking, besides being lineo-linear quadratics, must also be combinantive invariants to f and ϕ, remaining (as forms) unaltered for any linear substitutions impressed either upon the variables or upon the functions containing the variables.

Art. (67.). I must here premise that if there be any two forms of the same degree (and that degree odd) in x and y, a combinant may be formed from them, which will be linear in respect to each set of coefficients*. Thus calling the two functions

$$a_0.x^{2n+1}+(2n+1)a_1.x^{2n}.y+(2n+1).\frac{2n}{2}a_2.x^{2n-1}.y^2+\ldots+a_{2n+1}.y^{2n+1}$$

$$\alpha_0x^{2n+1}+(2n+1)\alpha_1.x^{2n}.y+(2n+1)\frac{2n}{2}\alpha_2.x^{2n-1}.y^2+\ldots+\alpha_{2n+1}.y^{2n+1},$$

the lineo-linear combinant in question will be

$$T=\left\{a_0.\alpha_{2n+1}-(2n+1).a_1.\alpha_{2n}+(2n+1)2n\ a_2.\alpha_{2n-1}+\frac{(2n+1)(2n)(2n-1)}{1.2.3}a_3\alpha_{2n-2}\ \&c.\ -a_{2n+1}.\alpha_0\ \&c.\right\}$$

which, using our customary notation, will be of the form

$$(0,\ 2n+1)-(2n+1)(1,\ 2n)+\frac{(2n+1)2n}{1.2}(2,\ 2n-1)\pm\&c.+(-)^n.\frac{(2n+1)(2n)(2n-1)\ldots(n+2)}{1.2.3\ldots n}(n,\ n+1).$$

As a corollary to this proposition (which, as well as the proposition itself, will be needed for the purposes of the ensuing determination), taking any function of an even degree in x, y, $F(x, y)$, there will exist a combinant to $\frac{dF}{dx}$ and $\frac{dF}{dy}$, by virtue of what has been stated above, which will be Mr. CAYLEY's well-known quadrivariant to F; viz. if $F=a_0.x^{2n}+a_1.x^{2n-1}+\ldots+a_{2n}.x^{2n}$, this will be

$$a_0.a_{2n}-2na_1.a_{2n}+\frac{2n(2n-1)}{2}a_2.a_{2n-2}+\ldots+\frac{1}{2}(-)^n.\frac{2n(2n-1)\ldots(n+1)}{1.2\ldots n}a_n^2.$$

The proposition itself is easily proved; first, the expression T being expressed entirely in terms of quantities of the form (r, s) remains unaltered for linear substitutions impressed upon the forms f and ϕ; it remains then only to show that T satisfies the differential equations to T treated as a mere invariant, viz.—

* I may add here incidentally (although not wanted for our present purposes) that as a combinant in which each set of coefficients enters linearly can always be formed to a system of functions 2 in number of as many variables and of any odd degree, so reciprocally can a combinant in which each set of coefficients enters linearly be always formed to a system of functions each of the degree 2, of which and of the variables contained in them, the number is any odd integer.

$$\left\{\begin{array}{l} a_0.\frac{d}{da_1}+2a_1.\frac{d}{da_2}+3a_2.\frac{d}{da_3}+\ldots+(2n+1)a_{2n}.\frac{d}{da_{2n+1}} \\ +\alpha_0.\frac{d}{da_1}+2\alpha_1.\frac{d}{da_2}+3\alpha_2.\frac{d}{da_3}+\ldots+(2n+1)\alpha_{2n}.\frac{d}{da_{2n+1}} \end{array}\right\}\mathrm{T}=0,$$

and

$$\left\{\begin{array}{l} a_{2n+1}.\frac{d}{da_{2n}}+2a_{2n}.\frac{d}{da_{2n-1}}+\ldots+(2n+1)a_1.\frac{d}{da_0} \\ +\alpha_{2n+1}.\frac{d}{da_{2n}}+2\alpha_{2n}.\frac{d}{da_{2n-1}}+\ldots+(2n+1)\alpha_1.\frac{d}{da_0} \end{array}\right\}\mathrm{T}=0.$$

From the hemihedral symmetry of T, which only changes its sign when the order of the coefficients in f and φ is simultaneously reversed, it is obvious that one of these equations cannot be satisfied without the other being so too. Looking then exclusively at the first of them, we see that this is satisfied by virtue of the equations

$$\left\{a_0.\frac{d}{da_1}+(2n+1)\alpha_{2n}\frac{d}{da_{2n+1}}\right\}\mathrm{T}=0$$

$$\left\{2a_1.\frac{d}{da_2}+2n.\alpha_{2n-1}.\frac{d}{da_{2n}}\right\}\mathrm{T}=0$$

. . . . . .

. . . . . .

$$\left\{(2n+1)a_{2n}\frac{d}{da_{2n+1}}+\alpha_0.\frac{d}{da_1}\right\}\mathrm{T}=0.$$

Hence then the differential equations to T being satisfied proves that it is an invariant, and, as above observed, its form shows upon its face that it is a combinant.

Precisely in the same way it may be demonstrated, that to two functions each of the same even degree $(2m)$ as

$$a_0x^{2m}+2ma_1.x^{2m-1}y+\frac{2m.(2m-1)}{2}a_2.x^{2m-2}y^2+\ldots+a_{2m}.y^{2m}$$

and $\quad \alpha_0.x^{2m}+2m\alpha_1.x^{2m-1}.y+2m.\frac{(2m-1)}{2}\alpha_2.x^{2m-2}.y^2+\ldots+\alpha_{2m}.y^{2m}$

there will be a quantity

$$\mathrm{G}=a_0.\alpha_{2m}-2ma_1.\alpha_{2m-1}+\frac{2m.(2m-1)}{2}a_2.\alpha_{2m-2}\pm\&\mathrm{c.}\ -2m\alpha_1a_{2m-1}+\alpha_0.a_{2m},$$

which, although not a combinant, will satisfy the differential equations necessary to prove it to be an ordinary invariant to the two given functions.

Art. (68.). Now let us consider the three forms f, φ and the subsidiary form

$$f=a_0x^m+ma_1.x^{m-1}.y+\ldots+a_m.y^m$$

$$\varphi=b.x^m+mb_1.x^{m-1}.y+\ldots+b_m.y^m$$

$$\Omega=u_1.y^{m-1}-(m-1)u_2.y^{m-2}.x\pm\&\mathrm{c.}\ +(-)a_m^{m-1}x^{m-1},$$

where $u_1, u_2, \ldots u_m$ are to be treated as constants.

Make

$$\mathrm{E}_{2i+1}.f=\frac{1.2\ldots(2i+1)}{m(m-1)\ldots(m-2i)}\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^{2i+1}.f$$

$$\mathrm{E}_{2i+1}.\varphi=\frac{1.2\ldots(2i+1)}{m(m-1)\ldots(m-2i)}\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^{2i+1}.\varphi,$$

i being any integer such that $2i+1$ does not exceed m, and now consider $E_{2i+1}.f$, $E_{2i+1}.\varphi$ as two functions of the degree $2i+1$ in ξ, η (x and y being regarded as constants); and by virtue of the formula in the last article, form T_i, the lineo-linear combinant of $E_{2i+1}.f$ and $E_{2i+1}.\varphi$; T_i will then be lineo-linear in respect to the coefficients in f and φ, and of the degree $2\left(m-(2i+1)\right)$ in respect to x and y. Again, let

$$E_i.\Omega = \frac{1.2\ldots 2i}{m(m-1)\ldots(m-2i+1)}\cdot\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^{2i}.\Omega.$$

$E_{2i}.\Omega$ treated as a function of ξ and η of the degree $2i$ will furnish a quadrinvariant Q_i of the degree $2(m-1-2i)$ in respect of x and y, and quadratic in respect of the system $u_1, u_2, \ldots u_m$. We have thus two forms, T_i and Q_i, each of the same even degree $\left(2m-(2i+1)\right)$ in respect of x, y. Forming between these the lineo-linear invariant G_i, G_i will be a function lineo-linear in respect of the coefficients of f and φ, and quadratic in respect of the system $u_1, u_2, \ldots u_m$. Moreover, G_i will (by the general principle of successive concomitance) be an invariant in respect to the system f, φ, Ω, and combinantive in respect to f and φ. Thus then G_i for all admissible values of i will belong to the family of forms to which the Bezoutiant is to be referred.

It requires to be noticed, that when i is taken (0), so that T_i and G_i are of the degree $2(m-1)$, E_i for this case must be taken equal to Ω^2, which evidently fulfills the required conditions of being of the degree $2(m-1)$ in (x, y), and quadratic in respect of the coefficients of Ω. If, now, m be even, we may take for $2i+1$ successively all the odd numbers from 1 to $(m-1)$ inclusively, and there will be $\frac{m}{2}$ forms G_i; when m is odd we may take for $2i+1$ successively all the odd numbers from 1 to m, and the number of forms of G_i will be $\frac{m+1}{2}$. It should be observed, that when m is odd and $2i+1=m$, T_i will become identical with the lineo-linear combinant to f and φ and Q_i with the quadrinvariant to Ω; and no power of x or y will enter into either, so that G_m will become simply $T_m \times Q_m$. I am now able to enunciate the proposition, that $G_0, G_1, \ldots G_{\frac{m}{2}-1}$, when m is even, and $G_0, G_1, \ldots G_{\frac{m-1}{2}}$, when m is odd, form the constituent scale of forms, of which the Bezoutiant and all other lineo-linear quadratic functions of m variables, which are combinants of the system f, φ, will be numerically-linear functions. I propose to term the members of this scale Co-bezoutiants.

As regards the present memoir, I shall content myself with exhibiting a partial verification of this law as regards the connection of the Bezoutiant with the G scale of Co-bezoutiants, and a complete determination of the numerical multipliers which express this connection for the cases comprised between $m=2$ and $m=6$ taken inclusively. It is impossible to predict for what ulterior purposes in the development of the Calculus of Invariants these numbers may or may not be required, and it seems

to me desirable that a commencement of a table containing them should be made and placed on record. The remaining pages of this memoir will accordingly be devoted to the ascertainment of them.

The theory of the Bezoutoid being included within that of the Bezoutiant, need not hereafter call for any special attention; I may merely notice that the Bezoutoid to a function of the degree (m) will be a numerico-linear function of $\frac{m-3}{2}$ of the G's if m be odd, and $\frac{m-4}{2}$ of the G's if m be even.

It will be more convenient hereafter to denote the G's as G_1, G_3, G_5 respectively, in lieu of G_0, G_1, G_2, &c., and to continue at the same time to give to the T's and Q's the same subscripts as the corresponding G's.

Art. (69.). 1st. Suppose $m=2$,

$$f=ax^2+2bxy+cy^2$$
$$\varphi=\alpha x^2+2\beta xy+\gamma y^2$$
$$\Omega=u_1.y-u_2.x.$$

Then

$$E_1.f=(ax+by)\xi+(bx+cy)\eta$$
$$E_1.\varphi=(\alpha x+\beta y)\xi+(\beta x+\gamma y)\eta$$
$$T_1=(ax+by)(\beta x+\gamma y)-(bx+cy)(\alpha x+\beta y)$$
$$=(a\beta-b\alpha)x^2+(a\gamma-c\alpha)xy+(b\gamma-c\beta)y^2$$
$$Q_1=\Omega^2=u_1^2y^2-2u_1.u_2xy+u_2^2.x^2$$

and
$$\therefore\ G_1=(a\beta-b\alpha)u_1^2+(a\gamma-c\alpha)u_1u_2+(b\gamma-c\beta)u_2^2.$$

Let us now form in the usual manner the Bezoutiant to f, φ; this is the quadratic function which corresponds to the matrix

$$\left.\begin{matrix}(2a\beta-2b\alpha)\,; & (a\gamma-c\alpha)\\ (a\gamma-c\alpha)\,; & (2b\gamma-c\beta)\end{matrix}\right\},$$

i.e. $\frac{1}{2}B=(a\beta-b\alpha)u_1^2+(a\gamma-c\alpha)u_1.u_2+(b\gamma-c\beta)u_2^2=G_1$ or $B=2G_1$.

2nd. Suppose $m=3$.

$$f=ax^3+3bx^2y+3cxy^2+dy^3$$
$$\varphi=\alpha x^3+3\beta x^2y+3\gamma xy^2+\delta y^3$$
$$\Omega=u_1y^3-2u_2.yx+u_3.x^3.$$

We have then

$$E_1.(f)=(ax^2+2bxy+cy^2)\xi+(bx^2+2cxy+dy^2)\eta$$
$$E_1.(\varphi)=(\alpha x^2+2\beta xy+\gamma y^2)\xi+(\beta x^2+2\gamma xy+\delta y^2)\eta$$
$$T_1=(ax^2+2bxy+cy^2)(\beta x^2+2\gamma xy+\delta y^2)-(bx^2+2cxy+dy^2)(\alpha x^2+2\beta xy+\gamma y^2)$$
$$=(a\beta-b\alpha)x^4+2(a\gamma-c\alpha)x^3y+\big(3(\beta\gamma-c\beta)+(a\delta-d\alpha)\big)x^2y^2+2(b\delta-d\beta)x^3y^3+(c\delta-d\gamma)y^4$$
$$Q_1=\Omega^2=u_1^2y^4-4u_1.u_2.y^3u+(4u_2^2+2u_1.u_3)y^2x^2-4u_2.u_3yx^3+u_3^2.x^4.$$

Supplying for facility of computation the reciprocals of the binomial coefficients to the index 4, viz.—

$$1\,;\quad -\tfrac{1}{4}\,;\quad \tfrac{1}{6}\,;\quad -\tfrac{1}{4}\,;\quad 1,$$

we obtain

$$G_1=(a\beta-b\alpha)u_1^2+2(a\gamma-c\alpha)u_1.u_2+\Big(2(b\gamma-c\beta)+\tfrac{2}{3}(a\delta-d\alpha)\Big)u_2^2$$

$$+\Big((b\gamma-c\beta)+\tfrac{1}{3}(a\delta-d\alpha)\Big)u_1.u_3+2(b\delta-d\beta)u_2.u_3+(c\delta-d\gamma)u_3^2.$$

It will here and henceforth be more useful to employ $[r, s]$ to denote, not the difference of the cross products of the $(r+1)$th and $(s+1)$th *entire* coefficients in f and φ, but the difference of the cross products of these coefficients divided each by its appropriate binomial coefficient. We may then write

$$G_1=[0, 1]u_1^2+2[0, 2]u_1.u_2+([1, 2]+\tfrac{1}{3}[0, 3])u_1.u_3+(2[1, 2]+\tfrac{2}{3}[0, 3]).u_2^2$$
$$+2[1, 3]u_2.u_3+[2, 3]u_3^2.$$

Again,

$$G_2=\{(a\delta-d\alpha)-3(b\gamma-c\beta)\}+(u_1.u_3-u_2^2)=([0, 3]-3[1, 2])(u_1u_3)-([0, 3]-3[1, 2])u_2^2.$$

Hence

$$G_1-\tfrac{1}{3}G_2=[0, 1]u_1^2+2[0, 2]u\ .u_3+2[1, 2]u_1.u_3+([0, 3]+[1, 2])u_2^2+2[1, 3]u_2.u_3+[2, 3]u_3^2.$$

But, again, the Bezoutiant of f, φ corresponds to the matrix

$3[0, 1]\,;$	$3[0, 2]\,;$	$[0, 3]$
$3[0, 2]\,;$	$[0, 3]+9[1, 2]\,;$	$3[1, 3]$
$[0, 3]\,;$	$3[1, 3]\,;$	$[3, 4].$

Hence summing the sinister bands to form the coefficients, we have

$$B=3[0, 1]u_1^2+6[0, 2]u_1^2.u_2+(3[0, 3]+9[1, 2])u_2^2+6[1, 3]u_2.u_3+[2, 3]u_3^2=3G_1-G_2.$$

3rd. Suppose $m=4$,

$$f=ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4$$
$$\varphi=\alpha x^4+4\beta x^3y+6\gamma x^2y^2+4\delta xy^3+\epsilon y^4$$
$$\Omega=u_1y^3-3u_2y^2x+3u_3yx^2-u_4.x^3.$$

Then

$$E_3.f=(ax+by)\xi^3+3(bx+cy)\xi^2\eta+3(cx+dy)\xi\eta^2+(dx+ey)\eta^3,$$

$$\therefore\quad T_3.f=\begin{Bmatrix}(ax+by)(\delta x+\epsilon y)\\ -(\alpha x+\beta y)(dx+ey)\end{Bmatrix}-3\begin{Bmatrix}(bx+cy)(\gamma x+\delta y)\\ -(\beta x+\gamma y)(cx+dy)\end{Bmatrix}$$
$$=([0, 3]-3[1, 2])x^3+([0, 4]-2[1, 3])xy+([1, 4]-3[2, 3])y^3$$

and

$$Q_2=(u_1.y-u_2x)(u_2y-u_3x)-(u_2y-u_3x)^2$$
$$=(u_1.u_3-u_2^2)y^2-(u_1.u_4-u_2u_3)xy+(u_2.u_4-u_3^2)x^2.$$

Hence supplying the binomial reciprocals

$$1 \quad ; \quad -\frac{1}{2} \quad ; \quad 1,$$

we have

$$G_3=([0,3]-3[1,2])(u_1.u_3-u_2^2)+\frac{1}{2}([0,4]-2[1,3])(u_1.u_4-u_2.u_3)$$
$$+([1,4]-3[2,3])(u_2.u_4-u_3^2).$$

Again,

$$T_3=(ax^3+3bx^2y+3cxy^2+dy^3)(\beta x^3+3\gamma x^2y+3\delta xy^2+\varepsilon y^3)$$
$$-(\alpha x^3+3\beta x^2y+3\gamma xy^2+\delta y^3)(bx^3+3cx^2y+3dxy^2+ey^3)$$
$$=[0,1]x^6+3[0,2]x^5y+(3[0,3]+6[1,2])x^4y^2+([0,4]+8[1,3])x^3y^3$$
$$+(3[1,4]+6[2,3])x^2y^4+3[2,4]xy^5+[3,4]y^6,$$

and $Q_1=\Omega^2$

$$=u_1^2.y^6-6u_1.u_2.x^5y+(9u_2^2+6u_1.u_3)y^4x^2-(2u_1.u_4+18u_2.u_3)x^3y^3$$
$$+(9u_3^2+6u_2.u_4)y^2x^4-6u_3.u_4.yx^5+u_4^2.x^6.$$

Hence, supplying the reciprocal binomial coefficients,

$$1 \quad ; \quad -\frac{1}{6} \quad ; \quad +\frac{1}{15} \quad ; \quad -\frac{1}{20} \quad ; \quad \frac{1}{15} \quad ; \quad -\frac{1}{6} ; \quad 1,$$

we find

$$G_1=[0,1]u_1^3+3[0,2]u_1.u_2+\left(\frac{1}{5}[0,3]+\frac{2}{5}[1,2]\right)(9u_2^2+6u_1.u_3)$$
$$+\left(\frac{1}{10}[0,4]+\frac{8}{10}[1,3]\right)(u_1.u_4+9u_2.u_3)+\left(\frac{1}{5}[1,4]+\frac{2}{5}[2,3]\right)\times(9u_3^2+6u_2.u^4)$$
$$+3[2,4]u_3.u_4+[3,4]u_4^2.$$

Now the Bezoutic square, taking account of the binomial factors in f and φ, may be written under the form

$$\begin{array}{cccc}
4[0,1]; & 6[0,2]; & 4[0,3]; & [0,4] \\
6[0,2]; & \left[\begin{array}{c}4[0,3]\\+24[1,2]\end{array}\right]; & \left[\begin{array}{c}[0,4]\\+16[1,3]\end{array}\right]; & 4[1,4] \\
4[0,3]; & \left[\begin{array}{c}[0,4]\\+16[1,3]\end{array}\right]; & \left[\begin{array}{c}[1,4]\\+24[2,3]\end{array}\right]; & 6[2,4] \\
[0,4]; & 4[1,4]; & 6[2,4]; & [3,4].
\end{array}$$

Hence the Bezoutiant B becomes

$$4[0,1]u_1^2+12[0,2]u_1.u_2+(4[0,3]+24[1,2])u_2^2+2[0,4]u_1u_4$$
$$+(2[0,4]+32[1,3])u_2.u_3+8[1,4]u_2.u_4+([1,4]+24[2,3])u_3^2$$
$$+12[2,4]u_3.u_4+[3,4]u_4^2.$$

And we ought to have $B=cG_1+eG_2$, to satisfy which equation we must manifestly have $c=4$; to find (e), compare the coefficients of u_2^2, this gives

$$4[0,3]+24[1,2]=\frac{36}{5}[0,3]+\frac{72}{5}[1,2]+e(3[1,2]-[0,3]);$$

accordingly we ought to be able to satisfy the two equations

$$\frac{36}{5}-e=4 \quad \frac{72}{5}+3e=24,$$

each of which accordingly we find is satisfied by the equality $e=\frac{16}{5}$.

Substituting in the equation for B above written, we thus obtain

$$B=4G_1+\frac{16}{5}G_2,$$

which will be found to be identically true.

Art (70.). We may now see our way to a more concise mode of obtaining the numerical coefficients [by which they may in fact be computed and verified with comparatively little labour], connecting the Bezoutiant with the co-bezoutiant forms of the constituent scale. It will not fail to have been remarked, that throughout the preceding determinations I have presumed the truth of the formulæ which admits of an immediate verification, that for all values of m and ω we have the identical equation

$$\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^{\omega}\cdot\left\{c_0x^m+mc_1x^{m-1}y+m.\frac{m-1}{2}c_2x^{m-2}y^2+\ldots+mc_{m-1}yx^{m-1}+c_m.x^m\right\}$$
$$=\frac{(m.m-1)\ldots(m-\omega+1)}{1.2\ldots\omega}\left\{L_0.\xi^{\omega}+\omega L_1.\xi^{\omega-1}.\eta+\omega.\frac{\omega-1}{2}L_2.\xi^{\omega-2}.\eta^2+\ldots+L_{\omega}.\eta^{\omega}\right\},$$

where

$$\left.\begin{aligned}
L_0&=c_0.x^{m-\omega}+(m-\omega)c_1.x^{m-\omega-1}.y+(m-\omega)\frac{m-\omega-1}{2}c_2.x^{m-\omega-2}.y^2\ldots+c_{m-\omega}.y^m\\
L_1&=c_1.x^{m-\omega}+(m-\omega)c_2.x^{m-\omega-1}.y+(m-\omega)\frac{m-\omega-1}{2}c_3.x^{m-\omega-2}.y^2+\ldots+c_{m-\omega+1}.y^m\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
L_{\omega}&=c_{\omega}.x^{m-\omega}+(m-\omega)c_{\omega+1}x^{m-\omega-1}.y+(m-\omega)\frac{m-\omega-1}{2}c_{\omega+2}.x^{m-\omega-2}.y^2\ldots+c_m.y^m
\end{aligned}\right\}$$

Let us now proceed to determine by an abridged method the linear relations corresponding to the cases of $m=5$, $m=6$, and first for $m=5$.

Let

$$f=ax^5+5bx^4y+10cx^3y^2+16dx^2y^3+5exy^4+hy^5$$
$$\phi=\alpha x^5+5\beta x^4y+10\gamma x^3y^2+10\delta x^2y^3+5\iota xy^4+\eta y^5$$
$$\Omega=u_1.y^4-4u_2.y^3x+6u_3.y^2x^2-4u_4.yx^3+u_5.y^4.$$

In forming G_0, G_2, G_4, let us confine our attention to the terms u_1^2; $u_1.u_3$; $u_1.u_4$.

A comparison of the coefficients of these with those in the Bezoutiant (B) will be sufficient for assigning the three numerical quantities which connect B with G_1, G_2, G_3. I omit $u_1.u_2$, because G_1 is the only one of the G's for any value of (m) which contains u_1^2 or $u_1.u_2$, and in G_1 the terms containing u_1^2 and $u_1.u_2$ are

$$[0, 1]u_1^2+(m-1)[0, 2].u_1.u_2,$$

and the corresponding part of the Bezoutiant is

$$m[0, 1]u_1^2+m.(m-1)[0, 2]u_1.u_2;$$

so that if we write

$$B=c_1.G_1+c_2.G_2+c_3.G_3+\&c.,$$

the two terms u_1^2 and $u_1.u_2$ will only enable us to form one equation with the c's, viz. $c_1=m$. Again, instead of considering the entire coefficients of $u_1.u_3$ and $u_1.u_4$, it will be sufficient to take a single argument of either of these coefficients (in the forms to be compared), as for instance $[0, 3]$ and $[1, 3]$. Then c_1 being known, c_2, c_3 will be determined; but for the purposes of verification I shall furthermore compute the whole of the coefficient of $u_1.u_2$.

Accordingly [calculating the G system in reverse order] we have

$$\begin{aligned} G_3&=\{[0, 5]-5[1, 4]+10[2, 3]\}\{u_1.u_2-4u_2.u_4+.3u_3^2\}\\ &=\{[0, 5]-5[1, 4]+10[2, 3]\}u_1.u_3+\ldots \end{aligned}$$

$E_2.f=(ax^3+2bxy+cy^3)\xi^3+3(bx^3+2cxy+dy^3)\xi^2\eta+3(cx^3+2dxy+ey^3)\xi\eta^2+(dx^3+2exy+fy^3)\eta^3$;

$E_2.\varphi=$ &c. &c.;

$$\begin{aligned} \therefore\quad T_3&=\{(ax^3+2bxy+cy^3)(\delta x^3+2\varepsilon xy+\varkappa y^3)-(ax^3+2\beta xy+\gamma y^3)(dx^3+2exy+hy^3)\}\\ &\quad-\{3(bx^3+2cxy+dy^3)(\gamma x^3+2\delta xy+\varepsilon y^3)-(\beta x^3+2\gamma xy+\delta y^3)(cx^3+2dxy+ey^3)\}\\ &=[0, 3]x^4+(2[0, 4]+\ldots)x^3y+\{[0, 5]+[1, 4]-8[2, 3]\}x^2y^3+\&c. \end{aligned}$$

[The number -8 results from the calculation $1-3(4-1)=-8$.]

Again,

$$E_2\Omega=(u_1y^2-2u_2yx+u_3x^2)\xi^2-2(u_2y^2-2u_3.yx+u_4x^2)\xi\eta+(u_3y^2-2u_4.yx+u_5x^2)\eta^2,$$

$$\begin{aligned} \therefore\quad Q_3&=(u_1.y^2-2u_2yx+u_3x^2)(u_3y^2-2u_4yx+u_5x^2)-(u_2.y^2-2u_3.yx+u_4.x^2)^2\\ &=u_1.u_3.y^4-2u_1.u_4.y^3x+u_1.u_5y^2x^2+\&c., \end{aligned}$$

all the terms and parts of terms unexpressed being free of u_1, and therefore not necessary for our purpose. Hence supplying the reciprocal factors

$$1\ ;\ -\frac{1}{4}\ ;\ \frac{1}{6}\ ;\ \ldots,$$

we have

$$G_3=[0, 3]u_1.u_3+([0, 4]+)u_1.u_4+\frac{1}{6}\{[0, 5]+[1, 4]+[2, 3]\}u_1.u_5+\&c.$$

Again, expressing $E_1.f$ and $E_1.\varphi$ in the usual way, we obtain

$$T_1=(ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4)(\beta x^4+4\gamma x^3y+6\delta x^2y^2+4\epsilon xy^3+\eta y^4)$$
$$-(\alpha x^4+4\beta x^3y+6\gamma x^2y^2+4\delta xy^3+\epsilon y^4)(bx^4+4cx^3y+6dx^2y^2+4exy^3+hy^4)$$
$$=[0,1]x^8+4[0,2]x^7y+(6[0,3]+)x^6y^2+(4[0,4]+)x^5y^3+([0,5]$$
$$+15[1,4]+20[2,3])x^4y^4+\&c.$$

(where it may be observed that the numbers 15 and 20 in the coefficient of $x^4.y^4$ arise from the quantities 4^2-1; 6^2-4^2).

Again, $Q_1=\Omega^2=u_1^2.x^8+8u_1.u_2x^7y+12u_1.u_3x^6y^2-8u_1.u_4x^5y^3+2u_1.u_5.x^4y^4+\&c.$

Hence supplying the multipliers

$$1;\ \frac{-1}{8};\ \frac{1}{28};\ \frac{-1}{56};\ +\frac{1}{70};\ \&c.$$

we have

$$G_1=[0,1]u_1^8+4[0,2]u_1.u_2+\frac{18}{7}[0,3]u_1.u_3+\frac{4}{7}[0,4]u_1.u_4$$
$$+\frac{1}{35}([0,5]+15[1,4]+20[2,3])u_1.u_5.$$

Again, the Bezoutiant

$$B=5[0,1]u_1^3+2.10[0,2]u_1.u_2+2.10[0,3]u_1.u_3+2.5[0,4]u_1.u_4+2.[0,5]u_1.u_5+\&c.$$

Accordingly, if we write $B=c_1.G_1+c_2.G_2+c_3.G_3$, we have, as above remarked, $c_1=5$; and to determine c_2, c_3, we have, by comparing the coefficients of $u_1.u_2$, $u_1.u_4$ in B, G_1, G_2, G_3,

$$20=\frac{90}{7}+c_2$$
$$10=\frac{20}{7}+c_3.$$

These two equations, then, as it turns out, are not independent, but are satisfied simultaneously by

$$c_3=\frac{50}{7}.$$

Finally, equating the coefficients of the several arguments in $u_1.u_5$, we have

$$0=5\times\frac{1}{35}+\frac{50}{7}\times\frac{1}{6}+c_2 \text{ from the argument } [0,5]$$

$$0=5\times\frac{15}{35}+\frac{50}{7}\times\frac{1}{6}+5c_2 \text{ from the argument } [1,4]$$

$$0=5\times\frac{20}{35}+\frac{50}{7}\times\frac{8}{6}+10c_2 \text{ from the argument } [2,3].$$

The 1st of which equations gives

$$c_2=2-\frac{1}{7}-\frac{25}{21}=\frac{14}{21}=\frac{2}{3};$$

the 2nd gives

$$c_2=\frac{3}{7}+\frac{5}{21}=\frac{2}{3},$$

and the 3rd gives

$$c_5=\frac{20}{21}+\frac{2}{7}=\frac{2}{3}.$$

We have thus abundantly verified the accuracy of the calculation, and there results the relation

$$B=5G_1+\frac{50}{7}G_3+\frac{2}{3}G_5.$$

Lastly, let $m=6$,

$$f=ax^6+6bx^5y+15cx^4y^2+20dx^3y^3+15ex^2y^4+6hxy^5+ky^6$$

$$\varphi=\alpha x^6+6\beta x^5y+15\gamma x^4y^2+20\delta x^3y^3+15\varepsilon x^2y^4+6\eta xy+\lambda y^6$$

$$\Omega=u_1.y^5-5u_2.y^4x+10u_3.y^3x^2-10u_4y^2x^3+50yx^4-u_6.x^6.$$

I shall here confine myself to the determination of a single argument in each of the terms u_1^2; $u_1.u_2$; $u_1.u_3$; $u_1.u_4$; $u_1.u_5$; $u_1.u_6$; this will be ample for the purpose of verification, as the equation to be assigned is of the form

$$B=c_5.G_1+c_3.G_3+c_5.G_5.$$

The arguments which I select as the most simple, will be those expressed by the symbols (0, 1); (0, 2); (0, 3); (0, 4); (0, 5); (0, 6) respectively, then we have

$$T_3=(ax+by)(\eta x+\lambda y)\mp\&c.-(hx+ky)(\alpha x+\beta y)$$
$$=([0,5]+\ldots)x^3+([0,6]+\ldots)xy+(\ldots)y^3$$
$$Q_3=(u_1.y-u_2x)(u_5y-u_6x)\mp\&c.$$
$$=(u_1.u_5+\ldots)y^3-(u_1.u_6+\ldots)yx+(\ldots)x^3.$$

Hence supplying the binomial reciprocals

$$1;\quad -\frac{1}{2};\quad 1,$$

$$G_3=([0,5]+\ldots)u_1.u_5+\frac{1}{2}([0,6]+\ldots)u_1.u_6+\&c.$$

Again,

$$T_3=(ax^3+\ldots)(\delta x^3+3\varepsilon x^2y+3\eta xy^2+\lambda y^3)\mp\&c.-(dx^3+3ex^2y+3hxy^2+ky^3)(\alpha x^3+\ldots)$$
$$=([0,3]+\ldots)x^6+(3[0,4]+\ldots)x^5y+(3[0,5]+\ldots)x^4y^2+([0,6]+\ldots)x^3y^3+\&c.$$
$$Q_3=(u_1.y^3\mp\&c.)(u_3y^3\mp3u_4y^2+3u_5x^2-u_6x^3)-\&c.$$
$$=(u_1.u_3+\ldots)y^6-(3u_1.u_4+\ldots)y^5x+(3u_1.u_5+\ldots)y^4x^2-(u_1.u_6+\ldots)y^3x^3+\&c.,$$

and the reciprocal binomial multipliers will be

$$1;\quad \frac{-1}{6};\quad \frac{+1}{15};\quad \frac{-1}{20};\ \&c.$$

Hence

$$G_3=[0,3]u_1.u_3+\frac{3}{2}[0,4]u_1.u_4+\frac{3}{5}[0,5]u_1.u_5+\frac{1}{20}[0,6]u_1.u_6\ \&c.\ \&c.$$

Finally,

$$T_1=(ax^5+\&c.)(\beta x^5+5\gamma x^4y+10\delta x^3y^2+10\epsilon x^2y^3+5\eta xy^4+\lambda y^5)-\&c.$$
$$=([0,1]+\ldots)x^{10}+5([0,2]+\ldots)x^9y+(10[0,3]+\ldots)x^8y^2+(10[0,4]+\ldots)x^7y^3$$
$$+(5[0,5]+\ldots)x^6y^4+([0,6]+\ldots)x^5y^5+\&c.$$
$$Q_1=\Omega^2=u_1^2y^{10}+(10u_1.u_2+\ldots)y^9.x+(20u_1.u_3+\ldots)y^8x^2+(20u_1.u_4+)y^7x^3$$
$$+(10u_1.u_5+\ldots)y^6x^4+(2u_1.u_6+\ldots)y^5x^5+\&c.;$$

and supplying the numerical series

$$1;\quad -\frac{1}{10};\quad \frac{1}{45};\quad \frac{-1}{120};\quad \frac{1}{210};\quad \frac{-1}{252};\ \&c.,$$

we have

$$G_1=[0,1]u_1^2+5[0,2]u_1u_2+\frac{40}{9}[0,3]u_1u_2+\frac{5}{3}[0,4]\ u_1.u_4$$
$$+\frac{5}{21}[0,5]u_1.u_5+\frac{1}{126}[0,6]u_1.u_6+\&c.$$

Again, the Bezoutiant

$$=6[0,1]u_1^2+30[0,2]u_1.u_2+40[0,3]u_1.u_3+30[0,4]u_1.u_4$$
$$+12[0,5]u_1.u_5+2[0,6]u_1.u_6+\&c.\ \&c.=B.$$

Hence making

$$B=c_1.G_2+c_3.G_3+c_5.G_5,$$

from u_1^2 and $u_1.u_2$ we obtain respectively

$$c_1=6$$
$$5c_1=30;$$

hence from $u_1.u_3$ and $u_1.u_4$ we obtain respectively

$$\left.\begin{aligned}\frac{240}{9}+\ c_3&=40\\ \frac{30}{3}+\frac{3}{2}c_3&=30\end{aligned}\right\}\ \text{or } c_3=\frac{40}{3};$$

hence from $u_1.u_5$ and $u_1.u_6$ we obtain respectively

$$6\times\frac{5}{21}+\frac{40}{3}\cdot\frac{3}{5}+c_5=12,\ i.e.\ c_5=12-8-\frac{10}{7}=\frac{18}{7}$$

$$6\times\frac{1}{126}+\frac{40}{3}\cdot\frac{1}{20}+\frac{1}{2}c_5=2,\ i.e.\ \frac{1}{2}c_5=2-\frac{2}{3}-\frac{1}{21}=\frac{9}{7};$$

hence

$$c_5=\frac{18}{27},$$

and the equation sought for is

$$B=6G_1+\frac{40}{3}G_3+\frac{18}{7}G_5.$$

Art. (71.). The following table exhibits the relations between the Bezoutiant and

the correspondent system of Co-bezoutiants for all values of m between 1 and 6 under a synoptical form.

$$m=1 \quad B=G_1$$

$$m=2 \quad B=2G_1$$

$$m=3 \quad B=3G_1-4G_3$$

$$m=4 \quad B=4G_1+\frac{16}{5}G_3$$

$$m=5 \quad B=5G_1+\frac{50}{7}G_3+\frac{2}{3}G_5$$

$$m=6 \quad B=6G_1+\frac{40}{3}G_3+\frac{18}{7}G_4.$$

These series could if wanted be easily extended, and the calculation of the coefficients reduced to a mere mechanical procedure.

If we suppose m to be $2i$ or $2i-1$, we have the equation

$$B=c_1.G_1+c_3.G_3+\ldots+c_{2i-1}G_{2i-1};$$

and it appears from the foregoing instances that the comparison of the coefficients, either of u_1^2, or of $u_1.u_3$ on the two sides of the equation, will serve to give c_1 and c_1 (which is always m being known), c_3 may be found by a comparison of the coefficients either of $u_1.u_3$, or of $u_1.u_4$, and so on for $c_5 \ldots c_{2i-1}$; all the coefficients in the equation for B above given, thus admitting of being found separately and successively and in two modes, so that there is a check at each step upon the correctness of the computations: the only exception to this last remark is (when m is odd) for the last coefficient of which the above condensed method affords only a single determination. I need hardly add the remark, that in substituting x^{m-1}, $x^{m-2}.y$; $\ldots x.y^{m-2} \ldots y^{m-1}$ in place of u_1, u_2, $\ldots u_{m-1}$, u_m respectively, all the G's become (to a numerical factor *près*) identical with one another and with the Jacobian to the system $(f\varphi)$.

Art. (72.). The foregoing theory took its origin (as will have been readily imagined) in meditations growing out of the celebrated theorem of M. Sturm. There appear to be several directions in which a development or extension of the subject matter of that theorem may be sought for. Thus a theory may be constructed relative to a single function of one or more variables, viewed in all cases as representing a geometrical locus. In the limiting case, when this locus becomes a system of points in a right line, we have the theorem of Sturm; generally the theory will be that of contours. Or, again, a theory may be formed in which the number of functions is always kept equal to that of the variables. We have then a theory of discreet points corresponding to roots, the number of real ones of which comprised within given limits it is the object of such theory to determine. M. Hermite, in a memoir recently presented to the French Institute, appears to have made a valuable addition to the Sturmian theory extended in this direction, to which the beautiful researches of M. Cauchy and the joint labours of MM. Liouville and Sturm, with reference to

the disposition of the imaginary roots of equations appear to have led the way. Finally, the number of variables may be supposed to be arbitrarily increased, but made always inferior by a unit to the number of the functions in which they are contained, or which comes to the same thing, we may construct the theory of a system of homogeneous functions equal in number to the variables in them, which in its simplest case becomes the theory of Intercalations which has been here partially considered, and which (as has been shown) embraces (not as a particular case, but as an implied consequence and easily extricated result) the theorem of M. STURM.

London, June 25, 1853.

General and Concluding Supplement.

Art. (א.). The expressions given in art. (*n.*) for the partial quotients of the continued fraction represented by $\frac{\phi x}{fx}$, are restricted to the supposition of all these partial quotients (except the first) being linear in x; when the first partial quotient is linear the formula (B.) of that article continues applicable on replacing $(D_i\ h_\theta)$by 1. I was forcibly struck by the peculiarity of these formulæ not ceasing to be true in consequence of the first partial quotient being supposed non-linear; and reflecting upon this, I was soon led to perceive that all the partial quotients might be supposed to be arbitrary integral functions of x, and the formulæ would still continue to apply to any such of them as might happen to be linear, although, as it were, imbedded among a group of other non-linear partial quotients. From this it was but an easy step to perceive that the formulæ A and B must admit of extension to the representation of partial quotients of any form, and that the dimorphism of the representation of the linear partial quotients could only be a consequence of the equation in integers $u+\nu=1$ having two solutions $u=0$, $\nu=1$ and $u=1$, $\nu=0$. I now proceed to enunciate the very remarkable general theorem (or as it may perhaps not inappropriately be termed Algebraical Porism), by virtue of which any partial quotient of a given degree in x belonging to an infinite continued fraction, all of whose partial quotients are algebraical functions of x, may be expressed to a constant factor *près*, by means of the numerator and denominator (or if we please either one of these) of the convergent *immediately* antecedent to and of the numerator and denominator of *any* convergent *not* antecedent to the partial quotient which is to be determined.

Art. (ב.). *Theorem.* Let $Q_1, Q_2, \ldots Q_i\ Q_{i+1} \ldots Q_n$, &c., each of an arbitrary degree in x, be the n first partial quotients of an algebraical continued fraction; let Q_{i+1} be the partial quotient to be determined and of the given degree ω_i; let

$$\frac{1}{Q_1-}\ \frac{1}{Q_2-}\ \frac{1}{Q_3-}\cdots\frac{1}{Q_i}=\frac{\phi_i(x)}{f_i(x)}$$

and

$$\frac{1}{Q_1-}\ \frac{1}{Q_2-}\ \frac{1}{Q_3-}\ \frac{1}{Q_i-}\ \frac{1}{Q_{i+1}}\cdots\frac{1}{Q_n}=\frac{\Phi(x)}{F(x)};$$

let u and ν be any couple of integers of the $\omega_{i+1}+1$ couples which satisfy the equation $u+\nu=\omega_{i+1}$; then, as usual, denoting the product of the differences of each of one set

of terms from each of another set, by writing the former under the latter, and calling $\eta_1\ \eta_2 \ldots \eta_\mu$ the μ roots of $\Phi(x)$, and $h_1\ h_2 \ldots h_m$ the m roots of $\mathrm{F}(x)$, (Φ and F being supposed respectively of μ and m dimensions in x), and forming the disjunctive equations

$$\theta_1, \theta_2, \theta_3 \ldots, \theta_\mu = 1, 2, 3, \ldots\ldots \mu$$
$$t_1, t_2, t_3 \ldots, t_m = 1, 2, 3, \ldots\ldots m,$$

we have the following equation,

$$\mathrm{Q}_{i+1} = \mathrm{K}_{u,\nu} \times \Sigma \left\{ (\varphi\eta_{\theta_1} . \varphi\eta_{\theta_2} \ldots . \varphi\eta_{\theta_\nu})^{\omega} \times (fh_{t_1} . fh_{t_2} \ldots . fh_{t_u})^{\omega} \right.$$

$$\left. \times \frac{\begin{bmatrix} \eta_{\theta_1} & \eta_{\theta_2} & \ldots\ldots \eta_{\theta_\nu} \\ h_{t_{u+1}} & h_{t_{u+2}} & \ldots\ldots h_{t_m} \end{bmatrix} \times \begin{bmatrix} h_{t_1} & h_{t_2} & \ldots\ldots h_{t_u} \\ \eta_{\theta_{\nu+1}} & \eta_{\theta_{\nu+2}} & \ldots\ldots \eta_{\theta_\mu} \end{bmatrix}}{\begin{bmatrix} \eta_{\theta_1} & \eta_{\theta_2} & \ldots\ldots \eta_{\theta_\nu} \\ \eta_{\theta_{\nu+1}} & \eta_{\theta_{\nu+2}} & \ldots\ldots \eta_{\theta_\mu} \end{bmatrix} \times \begin{bmatrix} h_{t_1} & h_{t_2} & \ldots\ldots h_{t_u} \\ h_{t_{u+1}} & h_{t_{u+2}} & \ldots\ldots h_{t_m} \end{bmatrix}} \frac{\big((x-\eta_{\theta_1})(x-\eta_{\theta_2}) \ldots . (x-\eta_{\theta_\nu})\big)\big((x-h_{t_1})(x-h_{t_2}) \ldots . (x-h_{t_m})\big)}{\Big\}},$$

and moreover the different values of $\mathrm{K}_{u,\nu}$ depending upon the different modes of breaking up ω_i into two parts u and ν are all (to a numerical factor *près*) equal to one another. Thus then the theorem pointed at in art. (*p.*) is discovered, and the way laid open (by an unexpected channel) for a complete discussion of the theory of the singular cases which may occur in the expansion of any rational algebraical fraction under the form of a continued fraction.

Art. (ב.). In the above expression, if we suppose $\omega_i = 1$, we have $u = 1$ and $\nu = 0$, or $u = 0$ and $\nu = 1$, and remembering that

$$\begin{bmatrix} h \\ \eta_1\ \eta_2 \ldots . \eta_\mu \end{bmatrix} = \Phi h \text{ and } \begin{bmatrix} \eta \\ h_1\ h_2 \ldots . h_m \end{bmatrix} = \mathrm{F}h$$

$$\begin{bmatrix} h_{t_1} \\ h_{t_2} h_{t_3} \ldots h_{t_m} \end{bmatrix} = \mathrm{F}'h, \text{ and } \begin{bmatrix} \eta_{\theta_1} \\ \eta_{\theta_2} \ldots . \eta_{\theta_\mu} \end{bmatrix} = \Phi' h,$$

Q_{i+1} becomes by virtue of the general formula representable under either of the equivalent forms

$$\mathrm{K}_{0,1}\ \Sigma_\mu^1 \left\{ (\varphi . \eta_\theta)^{\omega} \frac{\mathrm{F}\eta_\theta}{\Phi'\eta_\theta} (x - \eta_\theta) \right\} \text{ and } \mathrm{K}_{1,0}\ \Sigma_m^1 \left\{ (fh_t)^{\omega} . \frac{\Phi h_t}{\mathrm{F}'h_t} (x - h_t) \right\},$$

$\mathrm{K}_{0,1}$ and $\mathrm{K}_{1,0}$ being either equal, or differing only in the sign agreeably to the formulæ A and B.

Art. (ד.). It may be worth while to notice, that, although (of course) these formulæ and the general formulæ of (art. ב.), when supposed converted into functions of x and of the coefficients of F and of Φ by the reduction, integration and summation of the symmetrical functions of the roots which enter into them remain universally valid, and subject to no cases of exception, yet antecedently to these processes being performed the formulæ as they stand may become illusory when any relations of equality exist between the roots of Φ *inter se*, or between the roots of F *inter se*. Thus in the case before us, if Φ have equal roots the formulæ commencing with $\mathrm{K}_{0,1}$ is illusory, and if F have equal roots the other of the two formulæ becomes illusory.

Let us take the second of these and suppose that $\mathrm{F}(x)$ has

$$k_1 \text{ roots } c_1,\ k_2 \text{ roots } c_2,\ \ldots . k_p \text{ roots } c_p,$$

we may pass to the actual case from any case where the roots are infinitesimally near to the actual roots of $F(k)$, and all infinitesimally different from one another. Moreover the choice of the infinitesimal variations being arbitrary, let the k_i roots c_i be replaced by a group of roots

$$c_i+\delta;\ c_i+\delta\varrho_i;\ c_i+\delta\varrho_i^2;\ \ldots.\ c_i+\delta\varrho_i^{k_i-1},$$

where ϱ_i is a prime root of the equation $\varrho_i^{k_i}=0$, and δ is an infinitesimal quantity, and suppose each of the other groups to be varied in an analogous manner. Then it may easily be shown from this that the one of the formulæ in question will become

$$K_1^i\Sigma_p^1 k_t \frac{\left(\frac{d}{dc}\right)^{k-1}\{(fc_i)^2(\Phi c_i)(x-c_i)\}}{\left(\frac{d}{dc_i}\right)^k Fc_t};$$

and similarly, the twin formula becomes

$$K_1^0\Sigma_\varpi^1 \varkappa_\theta \frac{\left(\frac{d}{d\gamma}\right)^{\varpi-1}\{(\varphi\gamma_\theta)^2(F\gamma_\theta)(x-\gamma_\theta)\}}{\left(\frac{d}{d\gamma_\theta}\right)^\varpi \Phi\gamma_\theta}*.$$

Corresponding modifications will admit of being made by aid of a like method in the general formulæ of art. (2.) upon a similar supposition as to equalities springing up between the roots of fx *per se* and of $\varphi(x)$ *per se*, or between the roots of fx and φx *inter se*.

Art. (3.). If in (art. 2.) we take $i=0$, the formula for Q_{i+1} will become

$$Q_1=K_{u,\nu}\frac{\begin{bmatrix}\eta_{\theta_1} & \eta_{\theta_2} & \ldots\eta_{\theta_\nu}\\ h_{t_{u+1}} & h_{t_{u+2}} & \ldots h_{t_m}\end{bmatrix}\times\begin{bmatrix}h_{t_1} & h_{t_2} & \ldots h_{t_u}\\ \eta_{\theta_{\nu+1}} & \eta_{\theta_{\nu+2}} & \ldots\eta_{\theta_\mu}\end{bmatrix}}{\begin{bmatrix}\eta_{\theta_1} & \eta_{\theta_2} & \ldots\eta_{\theta_\nu}\\ \eta_{\theta_{\nu+1}} & \eta_{\theta_{\nu+2}} & \ldots\eta_{\theta_\mu}\end{bmatrix}\times\begin{bmatrix}h_{t_1} & h_{t_2} & \ldots h_{t_u}\\ h_{t_{u+1}} & h_{t_{u+2}} & \ldots h_{t_m}\end{bmatrix}}\Big((x-\eta_{\theta_1})\ldots(x-\eta_{\theta_\nu})\Big)\Big((x-h_{t_1})\ldots(x-h_{t_u})\Big),$$

u and ν being any two integers whose sum is ω_1, which is identical (as it ought to be) with the expression virtually contained in the formulæ of Section II. for the syzygetic multiplier of $\Phi(x)$ in the syzygetic equation connecting Fx and Φx with their first residue when Φx is supposed to be ω_1 dimensions in x lower than Fx identical, *videlicet*, in other words, with the integer part of the algebraical fraction $\frac{F(x)}{\Phi(x)}$.

* For in general if ρ is a prime root of the equation $\rho^\omega=1$, and if fx have ω roots all equal to c and ψx is any other function of x and if δ is an infinitesimal quantity, then rejecting all powers of δ higher than the $(\omega-1)$th degree,

$$\frac{\psi(c+\delta)}{f'(c+\delta)}+\frac{\psi(c+\rho\delta)}{f'(c+\rho\delta)}+\frac{\psi(c+\rho^2\delta)}{f'(c+\rho^2\delta)}+\ldots+\frac{\psi(c+\rho^{\omega-1}\delta)}{f'(c+\rho^{\omega-1}\delta)}$$

$$=\frac{1}{\left(\frac{d}{dc}\right)^\omega fc.\,\delta^{\omega-1}}\left\{\psi(c+\delta)+\rho\psi(c+\rho\delta)+\rho^2\psi(c+\rho^2\delta)+\ldots+\rho^{\omega-1}\psi(c+\rho^{\omega-1}\delta)\right\}$$

$$=\frac{\left(\frac{d}{dc}\right)^{\omega-1}\psi c\omega\delta^{\omega-1}}{\left(\frac{d}{dc}\right)^\omega fc\delta^{\omega-1}}=\omega\frac{\left(\frac{d}{dc}\right)^{\omega-1}\psi c}{\left(\frac{d}{dc}\right)^\omega fc}.$$

Art. (١.). When $\Phi(x)=F'(x)$,

$$\frac{\Phi(h_1)\Phi(h_2)\ldots\Phi(h_{\omega_{i+1}})}{\begin{bmatrix} h_1 & h_2 & \ldots h_{\omega_{i+1}} \\ h_{1+\omega_{i+1}} & h_{2+\omega_{i+1}} \ldots & h_m \end{bmatrix}} \text{ becomes identical with } (-)^{\frac{1}{2}(\omega_{i+1}-1)\omega_{i+1}}\zeta(h_1 h_2 \ldots h_{\omega_{i+1}}),$$

and we may consequently (using an extreme term in the forms in the polymorphic scale of forms representing Q_{i+1}), write

$$Q_{i+1}=(-)^{\frac{1}{2}(\omega_{i+1}-1)\omega_{i+1}} K_{\theta,\,\omega_{i+1}} \Sigma\zeta(h_1 h_2 \ldots h_{\omega_{i+1}})(f_i h_1)^{\theta}(f_i h_2)^{\theta}\ldots(f_i h_{\omega_{i+1}})^{\theta}(x-h_1)(x-h_2)\ldots(x-h_{\omega_{i+1}}).$$

Art. (١.). The following observations will serve to complete the theory of the singular cases in the expansion of an algebraical continued fraction.

Preserving the notation of art. (٢.), let

$$\sigma_i=m-(\omega_1+\omega_2+\ldots.+\omega_{i-1}+1),$$

Then (calling the roots of Fx, $h_1 h_2 \ldots h_m$) the (i)th simplified residue to $\frac{\Phi x}{F(x)}$, in accordance with the general formulæ for the residues in the second section (for greater simplicity selecting an extreme term of the polymorphic scale), will be represented by

$$\Sigma\frac{\Phi h_1\,\Phi h_2\,\Phi h_3 \ldots \Phi(h_{\sigma_i})}{\begin{bmatrix} h_1 & h_2 & h_3 & \ldots h_{\sigma_i} \\ h_{1+\sigma_i} & h_{2+\sigma_i} & h_{3+\sigma_i} \ldots & h_m \end{bmatrix}}(x-h_1)(x-h_2)(x-h_3)\ldots.(x-h_{\sigma_i}),$$

which will be of the form $L_i x^{\sigma_i-\omega_i+1}+$ &c., all the terms containing powers of x superior to σ_i vanishing by the coefficients becoming zero. If in the above expression we should use σ_i' in lieu of σ_i, where σ_i' is σ_i diminished by any integer inferior to ω_i, we should get other forms of the same residue, but these will all be of higher dimensions in the roots or coefficients than the one just given, and in fact the forms thus obtained corresponding to the values σ_i, σ_i-1, σ_i-2, $\ldots.\sigma_i-\omega_i+1$ substituted for σ_i in succession, would by aid of the relations of condition between the coefficients of Φx and Fx implied in the value of ω_i admit of being exhibited as a scale in which each form would be an exact algebraical product of the form which precedes it, multiplied by a function of the coefficients, and did space permit thereof it would be perfectly easy to give the forms of these multiplicators. But I pass on to the representation of what is more material, viz. the form of the complete residue in the case supposed, merely observing (as an *obiter dictum*) that the existence of each singular partial quotient (meaning thereby a quotient non-linear in x) only affects the form of the single simplified residue in immediate connexion with itself, and not at all the form of the other residues antecedent or subsequent to that one.

Art. (٣.). Let the ith simplified residue be called R_i and the corresponding complete residue $[R_i]$, then applying a method similar to the method given in Section I., we shall find that

$$(-)^{\gamma}[R_i]=\frac{L_{i-2}^{\omega_{i-1}+1}\cdot L_{i-2}^{\omega_{i-4}+1}\cdot \&c.}{L_{i-3}^{\omega_{i-1}+1}\cdot L_{i-3}^{\omega_{i-3}+1}\cdot \&c.}R_i,$$

L_i representing the leading coefficient in the (ith) simplified residue, and the sign of interrogation (?) denoting some function of $\omega_1 \omega_2 \ldots \omega_i$ (possibly a constant) remaining to be determined. And reverting to art. (ב.), the quantity that would be called $K_{\omega_1 \ldots \omega_i}$ according to the notation employed in the formulæ expressing Q_{i+1} in that article, will (abstraction being made of the algebraical sign and using for greater brevity (ι), ($\iota-1$), &c. to express $1+\omega_\iota$, $1+\omega_{\iota-1}$, &c.) come to be represented by

$$\frac{L_{i-1}^{\omega_i(i-1)}}{L_i^{(i)}} \cdot \frac{L_{i-2}^{4(i-3)} \cdot L_{i-3}^{4(i-3)}}{L_{i-2}^{4(i-2)} \cdot L_{i-3}^{4(i-4)}} \text{ &c.}$$

a similar convention being supposed to be made respecting the numerator and denominator of each convergent as was made respecting them in the particular case treated of in art. (f), page 473.

Art. (ט.). I will merely add a very few words in generalization of the method of limiting the roots of fx given in the Supplement to the fourth Section. As an inferior limit to fx is identical with a superior limit to $f(-x)$, we may confine our attention to superior limits alone. Suppose then that

$$\frac{\varphi x}{fx} = \frac{1}{Q_1-} \frac{1}{Q_2-} \cdots \frac{1}{Q_i-} \frac{1}{Q'_1-} \frac{1}{Q'_2-} \cdots \frac{1}{Q'_{i'}} \cdots\cdots \frac{1}{(Q)_1-} \frac{1}{(Q)_2-} \cdots \frac{1}{(Q)_i},$$

where the partial quotients Q are each of any arbitrary degree in x, and have all one algebraical sign in the coefficients of the highest powers of x from Q_1 to Q_i, and all the same sign (contrary to the former), in the coefficients of the highest powers of x from Q'_1 to $Q'_{i'}$, and so on alternately, then 1°, a superior limit to the superior limits of the cumulants $[Q_1 Q_2 \ldots Q_i]$, $[Q'_1, Q'_2 \ldots Q_{i'}]$, $\ldots [(Q)_1 (Q)_2 \ldots (Q)_{(i)}]$ will be a superior limit to fx, so that it remains only to give a rule for finding a superior limit to a cumulant $[Q_1, Q_2, Q_3 \ldots Q_i]$, which, 2°, is to be found by making

$$Q_1 - M_1 = 0,\ Q_2 - M_2 = 0,\ Q_3 - M_3 = 0 \ldots Q_i - M_i = 0,$$

where

$$M_1 = \mu_1 \quad M_2 = \mu_2 + \frac{1}{\mu_1} \quad M_3 = \mu_3 + \frac{1}{\mu_2} \ldots M_i = \frac{1}{\mu_{i-1}},$$

$\mu_1, \mu_2, \ldots \mu_{i-1}$ being any quantities entirely independent and arbitrary except in regard to their being all of the same sign as the leading coefficient in the element $Q_1, Q_2 \ldots Q_i$.

We may then find $L_1, L_2, \ldots L_i$ any superior limits to the roots of x in these i equations respectively; L, the greatest of these, will be a superior limit to the proposed cumulant $[Q_1 Q_2 \ldots Q_i]$; and it may be observed that $M_1 M_2 \ldots M_i$ are the general values which satisfy the equation

$$M_1 - \frac{1}{M_2-} \frac{1}{M_3-} \cdots \frac{1}{M_i} = 0,$$

subject to the condition that for all values of e

$$\frac{1}{M_e-} \frac{1}{M_{e-1}-} \frac{1}{M_{e-2}-} \cdots \frac{1}{M_1}$$

shall have a given invariable sign. The first part of the process, as just shown, consists in separating the type of the total cumulant which represents fx into partial

types, the point for each fracture of the total type being marked by a change of sign in the elements of the type for the value $x=+\infty$; it is easily seen therefore from this, that if $\frac{\Phi x}{Fx}$ is the generatrix of the cumulant in question, the number of such fractures (*i. e.* the number one less than the number of partial cumulants) will be the number of changes of algebraical sign in the signaletic series, consisting of the leading coefficients in Fx and in each of the odd-placed complete residues respectively, together with the number of changes of sign in the signaletic series, consisting of the leading coefficients in Φx and in each of the even-placed complete residues respectively.

The syzygetic theory of two algebraical functions, and the allied theory of algebraical continued fractions with their principal applications, may, I think, now be said to be completely made out, as well for the singular cases as for the general hypothesis.

Art. (').). I will conclude with observing that the theory within developed gives the means of transforming (explicitly and without the aid of symmetrical functions) into an algebraical continued fraction, any given sum of algebraical fractions of the form

$$\frac{c_1}{x-h_1}+\frac{c_2}{x-h_2}+\frac{c_3}{x-h_3}\cdots+\frac{c_n}{x-h_n},$$

where each c and h are supposed known. For let the above sum be called $\frac{\Phi x}{F(x)}$, then if h_θ, c_θ be used to denote any pair of corresponding terms of the h series and the c series, we have $\frac{\Phi h_\theta}{F'h_\theta}=c_\theta$, as is well known and easily proved. Again, if D_ix represent the simplified denominator of the ith convergent to the continued fraction equal to $\frac{\Phi x}{(Fx)}$ which is to be found, say

$$\frac{1}{(A_1x+B_1)-}\ \frac{1}{(A_2x+B_2)-}\cdots\frac{1}{A_nx+B_n},$$

we have $D_ix=\Sigma\dfrac{\begin{vmatrix}\Phi h_1 & \Phi h_2 \dots \Phi h_i\\ h_1 & h_2 \dots h_i\\ h_{i+1} & h_{i+2}\dots h_n\end{vmatrix}}{}(x-h_1)(x-h_2)\dots(x-h_n),$

$$=\Sigma(-)^{\frac{(i-1)}{2}\zeta}\frac{(h_1h_2\dots h_i)\Phi h_1\,\Phi h_2\dots\Phi h_i}{f'h_1.f'h_2\dots f'h_i}(x-h_1)(x-h_2)\dots(x-h_i)$$

$$=(-)^{\frac{i-1}{2}\zeta}\Sigma\{c_1c_2\dots c_i\zeta_0(h_1h_2\dots h_i)(x-h_1)(x-h_2)\dots(x-h_i)\}.$$

Therefore $(D_ih_1)^2=\{\Sigma(c_2c_3\dots c_{i+1})\zeta_0(h_2h_3\dots h_{i+1})(h_1-h_2)(h_1-h_3)\dots(h_1-h_{i+1})\}^2$

$$=\left\{\Sigma(c_2c_3\dots c_{i+1})\zeta^{\frac{1}{2}}(h_1h_2\dots h_{i+1})\zeta^{\frac{1}{2}}(h_2h_3\dots h_{i+1})\right\}^2;$$

and the simplified $(i+1)$th quotient, *i. e.* the value of $A_{i+1}x+B_{i+1}$, when divested of the allotrious factor, has been proved to be equal to

$$\Sigma(D_ih_1)^2\frac{\Phi h_1}{F'h_1}(x-h_1);$$

it is therefore now known as a rational and *integral* function of x; $h_1h_2\ldots h_n$; $c_1c_2\ldots c_n$. The allotrious factor itself is made up of the product of squares of quantities all of the same form as the leading coefficient in $D_x x$, which, from what has been shown above, is seen to be equal to

$$(-)^{\frac{i-1}{2}}\Sigma\{(c_1c_2\ldots c_i)\zeta(h_1h_2\ldots h_i)\}.$$

Hence each term in the continued fraction

$$\frac{1}{(A_1x+B_1)-}\ \frac{1}{(A_2x+B_2)-}\cdots\frac{1}{(A_nx+B_n)},$$

which is to be made equal to

$$\frac{c_1}{(x-h_1)}+\frac{c_2}{(x-h_2)}+\cdots\frac{c_n}{(x-h_n)},$$

is completely assigned in terms of x and the given quantities c and h.

Art. (2.). The number of effective intercalations between the roots of Φx, Fx is easily seen to be equal to the excess of the number of positive real numerators over the number of negative real numerators in the partial fractions of which $\frac{\Phi x}{Fx}$ is the sum, and hence we see *à priori*, as an obvious consequence of a simple extension of the reasoning in art. (47.), that the inertia of the quadratic function

$$\Sigma\left\{c_\theta(u_1+h_\theta u_2+h_\theta^2u_3+\ldots.+h^{n-1}.u_n)^2\frac{1}{x-h_\theta}\right\},$$

where $c_\theta=\frac{\Phi h_\theta}{Fh_\theta}$ will represent the value of the index in question. So too we may see that the formulæ given for the residues to $fx, f'x$ in art. (46.) continue to apply to the residues Fx, Φx. That is to say, these residues when divided out by Fx will be respectively represented by the successive principal coaxal determinants to the matrix

$$\begin{matrix}
S_0S_1\ S_2\ \ \ldots S_{m-1}\\
S_1S_2\ S_3\ \ \ldots S_m\\
S_2S_3S_3\ \ \ldots S_{m+1}\\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
S_{m-1}S_m\ S_{m+1}\ldots S_{2m-2},
\end{matrix}$$

where in general $S_r=\frac{c_1}{x-h_1}h_1^r+\frac{c_2}{x-h_2}h_2^r+\ldots+\frac{c_n}{x-h_n}h_n^r$;

and using the same matrix as above written with S′ substituted for S, where in general

$$S'_r=c_1(x-h)_1^r+c_2(x-h_2)h_2^r+\ldots+c_n(x-h_m)h_n^r;,$$

the successive principal coaxal determinants of the new matrix represent the successive denominators to the convergents of the continued fraction which expresses $\frac{\Phi x}{Fx}$.

The expression for the numerators to the convergents may also, there is no doubt, be obtained by some simple modification (dependent on introducing the quantities $c_1c_2\ldots c_n$) of the formula in art. (41.), p. 465.

I annex, more with the hope of suggesting than (in all instances) of conveying a

full conception of the force of the definitions, a Glossary, or rather a Repertory of the principal terms of art employed in the preceding pages, which might otherwise be apt to occasion some difficulty to persons unfamiliar with the subject.

ERRATA AND ADDENDA.

Page 408, 410, 412, 414, in running head to page, *for* Conjugate *read* Syzygetic.

—— 408, line 16 from foot, *for* above *read* about.

—— 409, line 4 from top, *for* continual *read* continued.

—— 429, line 12 from foot, *for* the same r new, *read* the same number r of new.

—— 430, line 3 from foot, *after* simplicity *insert* a comma.

—— 432, line 2 above (15.), *for* $\frac{1}{q'_{n-1}}$ *read* $\frac{1}{q_{n-1}}$.

—— 432, line 3 under (15.), *after* fraction *dele* —

—— 434, at end of the equation nearest the foot, *for* $\left(x \quad \eta_{q_1}\right)$ *read* $x-\eta_{q_1}$.

—— 436, in equation (21.), *for* $\left(x-\eta_{\xi_n}\right)$ *read* $\left(x-\eta_{\xi_r}\right)$.

—— 436, line 2 under (21.), *for* k_m *read* k_n.

—— 438, line 10 from foot, *for* $(\lambda_0)\ (\lambda_1)\ (\lambda_{n-1})$ *read* $\lambda_0\ \lambda_1\ \lambda_{n-1}$.

—— 439, line 3 from top, after the words "solution of" *insert* "the equation."

—— 439, line 10 from top, *for* and therefore *read* then.

—— 444, line 2 from top, *for* or *read* i. e.

—— 448, Art. 28, line 3, *for* $s-x^m$ *read* sx^m.

—— 452, line 1, *for* but *read* for.

—— 454, lines 5 and 14, *for* fm *read* f_n.

—— 458, line 4 in Art. 37, *for* fx *read* f'x.

—— 459, line 7 in Art. 38, *for* $-\mathrm{S}$ *read* $+\mathrm{S}$.

—— 464, line 15 from foot, *for* $k_1=k_3-k_5$ *read* $k_1=k_3=k_5$.

—— 467, line 6 from foot, *for* Latin and Greek *read* Latin, Greek and Hebrew.

—— 479, Art. p, line 2, *for* Σ'_m and Σ'_n *read* Σ^1_m and Σ^1_n.

—— 479, last line, *for* subscrolet *read* subscript.

—— 481, in the value of y_3 near foot of page, *for* the sign $-$ *read* $+$

—— 482, middle of the page, *for* $\frac{d^2f}{dx_2^2}$ *read* $\frac{d^2f}{dx_2^2}$.

—— 485, line 10, *for* ρ *read* x.

—— 497, Art. β, *for* Now *read* Also.

—— 504, line 12, *dele* $'\omega_1$.

—— 514, Art. (61.), lines 7 and 8, *for* $a_m.x^m$ and $b_m x^m$ *read* $a_m.y^m$, $b_m y^m$.

—— 515, line 4, *for* $u'_1, u'_2 \ldots u_m$ *read* $u'_1, u'_2 \ldots u'_m$.

—— 518, near middle of page, *for* $\frac{1}{1.2.3\ldots(m-1)^3}$ *read* $\frac{1}{(1.2.3\ldots(m-1))^3}$.

—— 524, near middle of page, *for* $\alpha_0.\frac{d}{da_1}$ *read* $\alpha_0.\frac{d}{da_1}$.

Glossary of new or unusual Terms, or of Terms used in a new or unusual sense in the preceding Memoir.

Allotrious.—The allotrious factor to a residue or quotient in the process of common measure applied to two algebraical functions is the constant factor of which such residue or quotient must be divested in order to become an integral and irreducible function.

Apocopated.—Applied to a type in the Theory of Cumulants, denotes a type the final or initial element of which has been taken away. If both are taken away, the type is said to be doubly apocopated.

Bezoutic.—For definition of Primary and Secondary Bezoutics see first Section. *Bezoutiant* to two functions, each of degree n, is a homogeneous quadratic invariantive function of n variables, the form of which serves to assign the index of the scale of the effective intercalations of the real roots of the two given functions.

Bezoutoid.—The Bezoutiant to two homogeneous functions obtained by differentiation from one homogeneous function of two variables. The Bezoutoid to a given function of m dimensions in the variables is accordingly a quadratic function of $(m-1)$ variables, the form of which is sufficient for determining the number of real roots in the given function.

Characteristic.—The employment of this word has been avoided in the preceding memoir; but as it contains an idea of capital importance in analysis, and especially in all inquiries of the kind here treated of, I subjoin the definition of its meaning. The characteristic of a simple condition of any kind is the rational integral function (in its lowest terms) whose evanescence necessarily and universally implies and is implied by the satisfaction of such condition. A simple condition has always a single characteristic, abstraction being made of the algebraical sign, which remains indeterminate. In like manner, a multiple condition, or a system of conditions, will have for its characteristic a plexus of rational integral functions, whose evanescence necessarily and universally implies and is implied by the satisfaction of such multiple condition or system of conditions. The number of functions in the characteristic plexus will however in general greatly exceed the index of the multiplicity of the conditions, and need not always be a unique system. There are however exceptions to this: thus the duplex condition, that a biquadratic function of x shall contain a cubic factor, or that a curve of the third degree shall have a cusp, will each be definitely characterized by a plexus of two functions, and no more.

The spirit of the higher analysis resides, and is to be sought for, in the *logic* of characteristics.

Co-bezoutiant.—Any homogeneous quadratic function similar in form and in its property of invariance to the Bezoutiant.

Cogredient and Contragredient.—A system of variables is cogredient to another system when it is subject to undergo simultaneously therewith linear substitutions of a like kind, and contragredient when it is subject to undergo linear substitutions simultaneously therewith but of a contrary kind.

Combinant.—A function of the quantities appearing in a given set of functions which remains unaltered as well for linear substitutions impressed upon the variables as for linear combinations of the functions themselves.

Concomitant.—*Nomen generalissimum* for a form invariantively connected with a given form or system of forms.

Conjunctive.—A syzygetic function of a given set of functions. Any function which universally,

and *subject to no cases of exception*, vanishes when a certain number of other functions all vanish together must be a conjunctive (*i. e.* a syzygetic function), or a root of a conjunctive of such functions. But if its vanishing is subject to cases of exception, then all that can be predicated of it is that it is *syzygetically related* to such functions, but it may, and usually does happen, that it will be syzygetically related to them in more than one way.

Contravariant.—A function which stands in the same relation to the primitive function from which it is derived as any of its linear transforms to an inversely derived transform of its primitive.

Covariant.—A function which stands in the same relation to the primitive function from which it is derived as any of its linear transforms to a similarly derived transform of its primitive.

Cumulant.—The denominator of the simple algebraical fraction which expresses the value of an improper continued fraction. See *Type*, infra.

Determinant.—This word is used throughout in the single sense, after which it denotes the alternate or hemihedral function the vanishing of which is the condition of the possibility of the coexistence of a system of a certain number of homogeneous linear equations of as many variables.

Dialytic.—If there be a system of functions containing in each term different combinations of the powers of the variables in number equal to the number of the functions, a resultant may be formed from these functions by, as it were, *dissolving* the relations which connect together the different combinations of the powers of the variables, and treating them as simple independent quantities linearly involved in the functions. The resultant so formed is called the Dialytic Resultant of the functions supposed; and any method by which the elimination between two or more equations can be made to depend on the formation of such a resultant is called a dialytic method of elimination. In such method accordingly the process of elimination between equations of a higher degree than the first is always reduced to a question of elimination between equations which are of the first degree only.

Discriminant.—The resultant of the n differential coefficients of a homogeneous function of n variables. See *Resultant*, infra.

Disjunctive.—A disjunctive equation is a relation between two sets of quantities such that each one of either set is equal according to some unspecified order of connexion with some one of the other set.

Effective scale of intercalations is the series of the real roots of two functions of x written in order of magnitude after repeated processes of removing pairs of roots belonging to either the same function (when not separated by roots of the other function): the roots of the two functions follow each other alternately.

Effluent.—From every homogeneous function of any number of variables i of the degree mm', where mm' are any two integers, may be formed (as shown in the Calculus of Forms, Section I.) a covariantive function of the degree m and of μ variables [where μ is the number of permutations that can be obtained by dividing m' into i parts (zeros admissible)], in which all the coefficients are numerical multiples of the given coefficients; covariants so formed may be termed effluents of their primitive. An example of this occurs in the foot note to Section V. p. 522, where the quantity there called Q is a quadratic effluent of the Jacobian.

Element.—A simple component of the type to a cumulant. See Cumulant, *supra*.

Emanant.—The result of operating any number of times (suppose i times) upon a given homogeneous function of any number of variables $x, y, z \ldots t$ with the operative symbol

$$\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}+\ldots+t'\frac{d}{dt}\right),$$

is called the ith emanant of the function operated upon. Every emanant is a covariant to its primitive, the new variables $x', y', z', \ldots t'$ being cogradient with the variables $x, y, z \ldots t$ with which they are respectively associated. $E_{2i+i}f$, $E_{2i+i}\phi$, page 522, are emanants of f and ϕ. The process of emanation is one of incessant occurrence in the theory of invariants. When the order of the emanation is the same as the degree of the function (supposed to be rational and integral) from which the emanation proceeds, the form of the original function is reproduced in the final emanant, the names only of the variables being changed.

Endoscopic, Exoscopic.—When the coefficients of the functions concerned in any investigation are regarded as integral indecomposable monads, the method is called exoscopic, and endoscopic when the coefficients are treated with reference to their internal constitution as composed of roots or other elements.

In addition to the examples in the foot note to Section 1, these words have a marked and most important application in the theory of Invariants, especially of two variables.

Form.—Any function may be regarded as an *opus operatum*; the matter operated upon being the variables, and the substance of the operations being the form, which resides in the function as the soul in the body. A form is always common to an infinity of functions, but for greater brevity may be and frequently is called by the name of some specified function in which it is contained.

Fundamental.—The fundamental scale of a system of Invariants or Concomitants is a set of the same, whereof every other is a Rational Integral Function.

Hessian or *Hessean*, named after Dr. Otto Hesse, of Konigsberg (the worthy pupil of his illustrious master, Jacobi, but who, to the scandal of the mathematical world, remains still without a Chair in the University which he adorns with his presence and his name), is the Jacobian to the differential coefficients of a homogeneous function of any number of variables. It is to a Jacobian what a Bezoutoid is to a Bezoutiant, or a Discriminant to a Resultant.

Hyperdeterminants.—See Memoir of Mr. Cayley, Cambridge and Dublin Mathematical Journal, May 1845, and Crelle's Journal of about the same date.

Improper, continued fraction is a continued fraction differing only from an ordinary one in the circumstance of negative signs being substituted for positive signs to connect the terms.

Inertia.—The unchangeable number of integers in the excess of positive over negative signs which adheres to a quadratic form expressed as the sum of positive and negative squares, notwithstanding any real linear transformations impressed upon such form.

Intercalations.—The theory of intercalations is the theory of the relative distribution of the real roots, or point-roots, of two or more equations, but in this theory the number of roots mutually interposed is to be taken only with reference to the number 2 as a modulus.

Invariance.—The property (under prescribed or implied conditions) of remaining invariable.

Invariant.—A function of the coefficients of one or more forms which remains unaltered when these undergo suitable linear transformations.

Inverse.—The inverse to a given square matrix is formed by selecting in its turn each component of the given matrix, substituting unity in its place, making all the other components in the same line and column therewith zero, and finally writing the value of the determinant corresponding to the matrix thus modified in lieu of the selected component. If the determinant to the matrix be equal to unity, its second inverse, *i. e.* the inverse to its inverse, will be identical, term for term, with the original matrix.

Jacobian.—The Jacobian to n homogeneous functions of n variables is the determinant represented by the symmetrical collocation in a square of the n differential coefficients of each of the n functions.

Kenotheme.—A finite system of discrete points defined by one or more homogeneous equations in number one less than the number of variables contained therein.

Limiting Series.—One set of quantities whose extreme values are exterior to the extreme values of a second set is set to limit the latter.

Matrix.—A square or rectangular arrangement of terms in lines and columns.

Minor Determinant.—Any determinant retained represented by a square group of terms arbitrarily chosen out of a matrix is a minor determinant thereto. The simple terms of the matrix are the last minors, and of course if the matrix is a square, it will itself in its totality represent a single complete determinant.

Monotheme.—A line, or finite system of lines, defined by one or more homogeneous equations two less in number than the numbers of the variables contained therein.

Order.—The orders of a homogeneous function are the linear functions of the variables the least in number by aid of which the function admits of being expressed.

Persymmetrical.—A symmetrical matrix, in which all the terms in the diagonal bands transverse to the axis of symmetry are identical, is said to be persymmetrical. *Ex.* An addition table.

Quadrinvariant.—An invariant of which the terms are quadratic functions of the coefficients of the primitive.

Relation (simple and compound). Vide *Substitution*, infra.

Resultant.—The resultant of n homogeneous general functions of n variables is that function of their coefficients which, equalled to zero, expresses in the simplest terms the condition of the possibility of their coexistence.

Rhizoristic.—A rhizoristic series is a series of disconnected functions which serve to fix the number of real roots of a given function lying between any assigned limits.

Signaletic.—A signaletic or *Semaphoretic* series is a sequence of disjunctive terms, considered solely with reference to the algebraical signs of *plus* and *minus* which they respectively carry.

Singular.—A proper algebraical function of a given degree, n, in one variable in its most general form, will, in respect to that variable, be of the nth degree in the denominator and the $(n-1)$th degree in the numerator, and will admit of being represented by a continued algebraical fraction of n terms, all of them linear.

But for particular values of, or relations among, the coefficients entering into the given fraction this mode of representation fails, and the continued fraction, instead of consisting of linear terms

n in number, will consist of terms, some of them at least, non-linear, and fewer than n in number. These then are the *singular* cases (or cases of singularity) in the theory of the development of an algebraical fraction under the continued fraction form; and it will be seen that according to this definition the case of the development of any proper algebraical fraction in which the degree of the numerator is more than one unit below that of the denominator, belongs (strictly speaking) to the class of singular cases; and this view of the case supposed is perfectly correct and conformable to the analogies of the subject.

Substitution (linear, *similar* or *contrary*).—A linear substitution is said to be impressed upon a system of variables when each variable is replaced by a linear conjunctive of all the variables. The matrix formed by the coefficients of substitution arranged in regular order is called the Matrix of Substitution, and is of course a square. When two substitutions (impressed in two systems of variables) have the same matrix, they are said to be *similar* and *contrary* when their matrices are contrary, *i. e.* mutually inverse to each other. When two systems of variables are supposed to be subject to the condition that their substitutions are always similar or always contrary, they are said to be related or in simple relation, the relation being of cogredience in the one case and of contragredience in the other.

When a linear substitution is impressed upon a system of independent variables, a corresponding linear substitution is necessarily impressed at the same time upon every complete system of homogeneous combinations (*i. e.* products and powers and products of powers) of these variables, the matrix to which latter substitution will consist of terms which will be functions (depending upon the degree of the homogeneous combinations) of the terms of the matrix to the primitive substitution. This matrix may be termed a compound matrix, having the primitive matrix for its base.

If, now, two systems of independent variables are subject to be synchronously impressed with substitutions, the matrices to which (not being both of them simple matrices) have for their bases matrices which are either similar or contrary, these two systems will be said to be in *compound relation* of cogredience in the one case, and of contragredience in the other.

Syrrhizoristic.—A syrrhizoristic series is a series of disconnected functions which serve to determine the effective intercalations of the real roots of two functions lying between any assigned limits.

Syzygetic.—A syzygetic function or conjunctive of a number of given rational integral functions is the sum of these affected respectively with arbitrary functional multipliers, which are termed the syzygetic multipliers. When a syzygetic function of a given set of functions can be made to vanish, they are said to be syzygetically related.

Transform.—Equivalent to the French noun substantive "*transformée.*"

Type.—The type of a cumulant is the series of the simple elements (or quotients), arranged in a fixed order, of which the cumulant is composed.

Umbral.—The umbral notation is a notation according to which simple quantities are denoted by syllables, instead of by single letters (the composition of these syllables being governed by the mode in which the quantities which they express are obtained); and the single letters of such syllables are termed umbral quantities or *umbræ*.

Weight.—In this memoir (throughout the earlier sections) the weight of any quantity composed of the product of the coefficients of any given function or functions of x is used to denote the number of roots of x appertaining to the given function or functions which must be employed to express such quantity. More generally, when dealing with a system of homogeneous functions,

the *weight* of a quantity may be defined with respect to *any selected variable* therein as the sum of the weights in respect to such variable of the several coefficients of which the quantity is composed (the weight of each several coefficient meaning the index of the power of the selected variable in that term of the given function or functions which is affected with such coefficient). These two definitions of *weight* may be perfectly well reconciled with each other by understanding the weight of a quantity formed from the coefficients of a function or system of functions of x to mean the weight, in respect *to unity*, of such quantity when the given functions are treated as homogeneous functions of x and 1.

Zeta.—The symbol ζ (preceding a row of bracketed terms) is used to denote the product of the squared differences of the terms which it affects.

[ ]. A bracket of this form, when inclosing a superior and an inferior row of terms m and n in number respectively, indicates the mn products of the differences obtained by subtracting each term in the second row from each term in the first row; when enclosing an arrangement of terms in a single line, it is used to denote the cumulant of which such an arrangement is the type.

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XIX. The Bakerian Lecture.—*On the Influence of the Moon on the Magnetic Declination at Toronto, St. Helena, and Hobarton.* By *Colonel* Edward Sabine, *R.A., Treas. and V.P.*

Received and Read November 17, 1853.

THE success which attended the endeavour to detect the influence of the moon on the pressure of the atmosphere, by a suitable arrangement of the hourly barometrical observations at St. Helena*, naturally suggested the idea that the influence of the moon on the direction of the magnetic needle, supposing such an influence to exist, might be manifested by an analogous arrangement of the hourly magnetical observations at that station; inasmuch as the magnetical disturbances due to other causes, and liable to mask so small an effect as that which might be anticipated from the moon, were, like those of the barometer, of inconsiderable amount at St. Helena when compared with those at many other stations.

An examination of observations of the Declination made at Milan, whilst M. Kreil was Director of that observatory, led him, in a memoir read in February 1841 to the Royal Bohemian Society of Sciences, to announce his belief that the moon does actually exercise an influence on the magnetic direction at the surface of our globe, cognisable by a variation in the Declination depending on the moon's hour-angle, and completing its period in a lunar day. M. Kreil has since confirmed the discovery thus announced by investigations based on a more extensive series of similar observations made under his direction at Prague, and discussed,—1st, in the 'Magnetische und Meteorologische Beobachtungen zu Prag' for 1841; and 2nd, in a memoir presented to the Imperial Academy of Sciences at Vienna in June 1850, and published in the Transactions of that Academy in 1852.

Meanwhile, in the course of the discussion of the results derivable from the Observations at the British Colonial Observatories, I had selected for a primary examination of this subject the series of hourly observations of the Declination at St. Helena, extending from September 1842 to August 1847 inclusive; and having determined on the process through which the observations should be passed, the work of reduction was commenced, and though occasionally interrupted by more pressing duties, was resumed from time to time, and was at length completed for the five years in the early part of the summer of 1852. The result was conclusive, in so far as a variation depending upon the moon's hour-angle was systematically and consistently manifested; but this variation differed so considerably in many important particulars, as

* Philosophical Transactions, 1847, Art. V.

well as in amount, from the conclusions which M. Kreil had derived from the Milan and Prague observations, as to make it plain to perceive that a knowledge of the phenomena at many different parts of the globe must be obtained, before a general theory of the nature and character of the moon's magnetic influence could be arrived at by inductive reasoning. Having, therefore, in my possession a similar series of hourly observations at Toronto and Hobarton, which stations might with St. Helena be deemed to a certain extent representatives of the phenomena in the middle latitudes of the two hemispheres and in the tropics, I thought it best to retain the conclusions at St. Helena until they should be accompanied by those at Toronto and Hobarton. I took with me, however, to the Belfast Meeting of the British Association in September 1852 an abstract of the St. Helena results, and showed them to several persons interested in magnetical researches who attended that meeting.

The investigation has since been completed for the three stations, so far at least as regards the magnetic *Declination*, and the results form the subject of this communication; to be followed, should circumstances permit, by the results of a similar investigation into the corresponding phenomena of the Inclination and Total Force. It has appeared the most convenient arrangement to treat the stations separately, and to take them in the order of their succession from North to South; commencing therefore with Toronto.

Toronto.—The results at this station are obtained from six years of hourly observation, extending from July 1842 to June 1848 inclusive. The observations were received at Woolwich in the form of Monthly Returns, in which the scale-reading of the Declinometer at every hour of mean Göttingen Time was inserted in tables of double entry, having the hours in vertical columns and the days in horizontal lines, with the daily means in the last vertical column, and the monthly means at the different hours of solar time in the bottom horizontal line. The process at Woolwich commenced by marking for omission all those disturbed observations, whensoever occurring, which exceeded a certain limit on either side of the monthly mean in the same month and at the same hour. When a disturbance was so considerable as materially to affect the amount of the monthly mean at the bottom of the page, a fresh monthly mean was taken, omitting the disturbed observation and giving fresh limiting values, by which the disturbances were marked afresh. The limits adopted, beyond which disturbances were omitted and within which observations were retained for the proposed investigation, were (at Toronto) four scale divisions (or 2′·9 in arc) above or below the monthly mean at the same hour. The observations retained were then marked in small figures on the face of the returns with the lunar hour to which each observation most nearly corresponded. For this purpose the time of the moon's passage of the meridian at Greenwich was taken from the Nautical Almanac, and corrected for the difference of longitude, so as to give the time of the moon's passage of the astronomical meridian at Toronto in the mean solar time of the station. The difference of time corresponding to the difference between the meridians of Toronto and Göt-

tingen was then applied, so as to give the mean *Göttingen* time of the moon's passage of the astronomical meridian of Toronto. The observation at the Göttingen hour nearest to the time thus computed was then marked with 0^h, signifying that it was the nearest observation to the moon's upper culmination, and from which its distance could not exceed half an hour. The time of the moon's inferior passage was then computed in a similar manner, and the observation at the Göttingen hour nearest to it was marked 12^h. The intermediate hours received corresponding markings, except that on those occasions when thirteen solar hours, and consequently thirteen observations, were comprised within twelve lunar hours, that observation was omitted which fell most nearly equidistant between the epochs of two exact lunar hours. The observations were then considered to be prepared for arrangement in *lunar tables*, but instead of the observations themselves, the differences at each hour between the scale-reading observed and the mean scale-reading in the same month and at the same hour were entered in these tables, by which process the diurnal and other variations depending on the time of the solar year and the hour of the solar day were, in great part at least, eliminated. The differences were marked with a + or − sign according as the scale-reading at the time of observation was greater or less than the monthly mean at the same hour; the entries having a + sign implying a more westerly direction of the north end of the magnet than its mean direction in the same month and at the same hour; and those having a − sign implying the converse. The mean was then taken, in every month, of every lunar hour (taking the signs into account); the monthly means were collected into yearly means,—and finally, the means at each lunar hour in each of the six years of observation were collected in a single table.

The whole number of observations in the six years at Toronto which were thus treated amounted to 44,754, of which 6356 were put aside in consequence of the amount of disturbance (measured from the monthly mean at the same hour) equalling or exceeding $2'{\cdot}9$ in arc.

The following Table shows the amount of Easterly or Westerly deflection of the north end of the magnet at the different lunar hours derived from the observations of the six years by the process which has been described, and inferred to be due to the moon's action. The partial results, or those afforded by the separate years, will be printed in the introduction to the third volume of the Toronto Observations. They exhibit only such small partial differences from the mean of the six years as may reasonably be ascribed to the operation of other causes than the moon's influence; and present no notable trace of a progressive increase in the amount of the variation in the successive years from 1843 to 1847, such as that which has been found to characterize the variations depending upon the *solar* hours.

The deflections in the subjoined Table are in scale divisions, of which each $=0'{\cdot}721$. The readings to which the − signs are prefixed denote Easterly deflections, and those which have the + signs prefixed Westerly deflections.

TABLE I.

Lunar hours.	Deflections.	Lunar hours.	Deflections.	Lunar hours.	Deflections.	Lunar hours.	Deflections.
	Scale divisions.		Scale divisions.		Scale divisions.		Scale divisions.
21	−0·15	4	+0·14	9	−0·12	16	+0·03
22	−0·35	5	+0·175	10	−0·30	17	+0·17
23	−0·30	6	+0·17	11	−0·43	18	+0·31
0	−0·35	7	+0·11	12	−0·42	19	+0·11
1	−0·32	8	+0·01	13	−0·30	20	+0·06
2	−0·225			14	−0·19		
3	−0·14			15	−0·12		

We may represent these numbers by the general formula—

$$\begin{aligned}\Delta_x = &-0{\cdot}10125 + {\cdot}0147 \sin(a + 36^\circ\ 43') - {\cdot}2993 \sin(2a + 276^\circ\ 51') \\ &- {\cdot}0045 \sin(3a + 247^\circ\ 23') + {\cdot}0234 \sin(4a + 68^\circ\ 13') \\ &+ {\cdot}0376 \sin(5a + 49^\circ\ 19') - {\cdot}0195 \sin(6a + 16^\circ\ 08') \\ &- {\cdot}0135 \sin(7a + 345^\circ\ 25') - {\cdot}0109 \sin(8a + 6^\circ\ 35') \\ &- {\cdot}0266 \sin(9a + 307^\circ\ 22') + {\cdot}0244 \sin(10a + 58^\circ\ 17') \\ &+ {\cdot}0043 \sin(11a + 87^\circ\ 01') + {\cdot}00833 \cos 12a\,;\end{aligned}$$

a being the difference between x and 18 lunar hours expressed in hours and parts of an hour multiplied by 15°.

With this formula we obtain the times of greatest easterly deflection $0^h\ 22^m$ and $11^h\ 23^m$, and the amounts $15''{\cdot}7$ and $19''{\cdot}1$: and of greatest westerly deflection $5^h\ 0^m$ and $18^h\ 0^m$, and the amounts $7''{\cdot}6$ and $13''{\cdot}4$. If we suppose the differences between the times and amounts of the two extreme easterly deflections to be due, wholly or in part, to accidental irregularities in the amount of the deflections at the different hours, the exact values of which would probably require a longer period of observation to determine,—and if the same remark be also applicable to the times and amounts of the two extreme westerly deflections,—we find, as the mean of the times of extreme easterly deflection, $11^h\ 53^m{\cdot}5$, and $23^h\ 53^m{\cdot}5$, and the amount $17''{\cdot}4$; and of the extreme westerly deflections $5^h\ 30^m$, and $17^h\ 30^m$, and the amount $10''{\cdot}5$; making the total amount of the variation due to lunar influence $(17''{\cdot}4 + 10''{\cdot}5 =) 27''{\cdot}9$.

The progression being a double one in the 24 lunar hours the variation passes four times through zero when the lunar variation disappears or equals 0. These times as given by the formula are $3^h\ 29^m$; $8^h\ 08^m$; $15^h\ 45^m$; and $20^h\ 27^m$; or $3^h\ 33^m$ before, and $3^h\ 29^m$ after the moon's upper culmination; and $3^h\ 52^m$ before, and $3^h\ 45^m$ after the moon's lower culmination.

It is quite possible that when the facts shall have been more precisely determined, it may prove that neither the times nor the amounts of the two easterly deflections are strictly symmetrical; and so also in regard to the two extreme westerly deflections. With a full consideration of what may be due to accidental irregularities, and particularly to the influence on the precise time of the extreme elongation of even a very small error occurring near the turning hours, there still seems an indication of systematic difference, which is the more deserving of attention, because M. KREIL

has inferred the existence of an inequality of the same kind, especially in the amounts of the similar deflections at opposite points of the hour-circle, in the Prague observations.

The disparity in the amounts of the deflections in *opposite* directions, viz. 17″·4 easterly, and 10″·5 westerly, is one which, from its magnitude, cannot be imagined to be occasioned by accidental irregularities, and must be regarded as a real feature of the phenomena under discussion.

M. Kreil has inferred from the Prague observations, that the variation due to the moon's influence is not equally cognisable in all parts of the year, being greatest in summer, and disappearing wholly in the three winter months. To examine this question, I have caused in the following table the results of the Toronto observations to be arranged in two divisions, one comprising the results of thirty-six months composed of the six summer months in each of the six years under examination, and the other also of thirty-six months composed of the six winter months in each of the same years.

Table II.

Lunar hours.	Summer.	Winter.	Table I.	Lunar hours.	Summer.	Winter.	Table I.
	Scale divisions.	Scale divisions.	Scale divisions.		Scale divisions.	Scale divisions.	Scale divisions.
0	−0·51	−0·18	−0·35	12	−0·43	−0·42	−0·42
1	−0·54	−0·10	−0·32	13	−0·40	−0·21	−0·30
2	−0·49	+0·05	−0·225	14	−0·32	−0·08	−0·19
3	−0·35	+0·07	−0·14	15	−0·37	+0·12	−0·12
4	−0·08	+0·36	+0·14	16	−0·09	+0·15	+0·03
5	+0·07	+0·29	+0·175	17	−0·01	+0·31	+0·17
6	+0·13	+0·21	+0·17	18	+0·27	+0·34	+0·31
7	+0·14	+0·09	+0·11	19	+0·15	+0·07	+0·11
8	+0·11	−0·09	+0·01	20	+0·07	+0·04	+0·06
9	+0·01	−0·25	−0·12	21	−0·15	−0·15	−0·15
10	−0·26	−0·33	−0·30	22	−0·45	−0·24	−0·35
11	−0·42	−0·42	−0·43	23	−0·43	−0·18	−0·30

The principal differences which show themselves in this comparison of opposite seasons are, that the easterly deflection at the upper culmination is less, and the westerly deflection on both sides of the meridian is greater, in the winter than in the summer half-period; at the inferior culmination the amount of extreme easterly deflection is nearly identical in the two seasons. For the purpose of examining how far similar partial differences would be shown when periods of similar duration are taken without reference to season, I arranged as in the following table, a comparison of the results in two half-periods, each comprising three complete years (including both summer and winter months), and each, therefore, containing the same number of months, viz. thirty-six, as in Table II.

TABLE III.

Lunar hours.	October 1842 to September 1845.	October 1845 to September 1848.	Table I.	Lunar hours.	October 1842 to September 1845.	October 1845 to September 1848.	Table I.
	Scale divisions.	Scale divisions.	Scale divisions.		Scale divisions.	Scale divisions.	Scale divisions.
0	−0·32	−0·39	−0·35	12	−0·49	−0·35	−0·42
1	−0·26	−0·38	−0·32	13	−0·34	−0·26	−0·30
2	−0·18	−0·27	−0·225	14	−0·27	−0·12	−0·19
3	−0·05	−0·23	−0·14	15	−0·11	−0·13	−0·12
4	+0·15	+0·14	+0·14	16	+0·07	−0·01	+0·03
5	+0·21	+0·14	+0·175	17	+0·20	+0·10	+0·17
6	+0·20	+0·13	+0·17	18	+0·31	+0·31	+0·31
7	+0·15	+0·08	+0·11	19	+0·13	+0·09	+0·11
8	−0·04	+0·05	+0·01	20	+0·07	+0·04	+0·06
9	−0·10	−0·14	−0·12	21	−0·17	−0·13	−0·15
10	−0·34	−0·26	−0·30	22	−0·31	−0·39	−0·35
11	−0·47	−0·37	−0·43	23	−0·39	−0·22	−0·30

The inference from these two comparisons, Tables II. and III., appears to be simply, that in consequence of the small amount of the moon's influence, which has to be brought out from amongst many disturbing causes of much greater influence, the mean result of six years is considerably more satisfactory than the mean results of three years of observation, whether the latter consist of three complete years, or of three years composed either of thirty-six summer or of thirty-six winter months; but that the variation due to the moon's influence is distinctly cognisable in all cases. The results may be more irregular in the winter months, in consequence possibly of the greater prevalence of the other disturbing causes at that season; and these, in a less complete system of observation, may in the extreme winter months, as appears to be the case at Prague, even wholly mask the effect of the moon's influence. In the present case, if we take a mean of opposite points of the hour-circle at Toronto in the summer and winter half-period, we find the amount of the whole deflection, between 0^h and 12^h on the one hand, and 6^h and 18^h on the other hand, to be $29''$ in the three years consisting of thirty-six summer months, and $24''{\cdot}9$ in the three years consisting of thirty-six winter months. If we do the same by the half-periods, each of three complete years or of eighteen summer and eighteen winter months, we find the whole deflection between 0^h and 12^h on the one hand, and 6^h and 18^h on the other, $28''{\cdot}5$ in the years 1842–45, and $24''{\cdot}4$ in the years 1845–48. The differences from the results of the six years combined are nearly equal in both cases.

In order to examine whether any difference is to be traced in the lunar diurnal variation which it might be possible to attribute to the moon's change of declination, I caused two tables to be formed, consisting respectively of the lunar diurnal variation on the three days of highest northern, and on the three days of highest southern declination throughout the six years of observation; the three days in the first table consisted in every case of the day when the moon was at her extreme distance north of the equator, and of one day before and one day after; and in the second table of the day when she was at the extreme distance south of the equator, and of

one day on either side of that day. The first table comprised 230 days and the second 226 days; the reason of the number of days falling short of the full complement being the intervention of Sundays, when no observations were recorded. Table IV. exhibits in one view the results of these two tables. It does not appear to present any satisfactory evidence of the existence of a difference either in the character or the amount of the lunar diurnal variation at the periods of the moon's extreme divergence on either side of the equator; at the same time the number of days (230 and 226) is too few to furnish *positive* evidence as to the existence of a *small* difference; the general character of the variation appears to be the same in both cases, when allowance is made for irregularities which must be expected to present themselves in such short periods of comparison.

TABLE IV.

Lunar hours.	Days of the moon's extreme north declination.	Days of the moon's extreme south declination.	Mean of six years, Table I.	Lunar hours.	Days of the moon's extreme north declination.	Days of the moon's extreme south declination.	Mean of six years, Table I.
	Scale divisions.	Scale divisions.	Scale divisions.		Scale divisions.	Scale divisions.	Scale divisions.
0	−·35	−·35	−·35	12	−·47	−·31	−·42
1	−·33	−·17	−·32	13	−·14	−·28	−·30
2	−·48	−·10	−·225	14	−·16	−·28	−·19
3	−·13	+·13	−·14	15	−·16	−·10	−·12
4	+·05	+·11	+·14	16	+·07	−·04	+·03
5	+·18	−·04	+·175	17	+·01	+·36	+·17
6	+·15	+·01	+·17	18	+·15	+·18	+·31
7	+·16	−·03	+·11	19	−·12	+·19	+·11
8	−·01	+·10	+·01	20	+·14	+·08	+·06
9	·00	+·07	−·12	21	+·02	+·04	−·15
10	−·19	−·16	−·30	22	−·04	−·35	−·35
11	−·43	−·25	−·43	23	−·22	−·38	−·30

St. Helena.—The hourly observations at St. Helena, which have been employed in this investigation, extended from September 1842 to August 1847, comprehending five years. They have been treated in the manner already described in the discussion of the Toronto observations, with this difference only, that as the magnetic disturbances are less in amount at St. Helena, the limits of exclusion on account of disturbance were taken at 2·5 scale divisions, or 1′·78 in arc.

The whole number of observations at St. Helena, which have been thus treated, amounts to 36,449, of which 2678 have been put aside in consequence of the amount of disturbance (measured from the monthly mean at the same hour) equalling or exceeding 1′·78 in arc. The following Table exhibits the mean variation in the five years at the several lunar hours, due to the moon's influence. The deflections are those of the north end of the magnet, and are easterly when the − sign is used, and westerly when the + sign is used. The value of a scale division at St. Helena was 0′·711.

TABLE V.

Lunar hours.	Deflections.	Lunar hours.	Deflections.	Opposite points on the hour-circle combined.	
				Lunar hours.	Deflections.
	Scale divisions.		Scale divisions.		Scale divisions.
2	·00	14	·00	2 and 14	·00
3	−·10	15	−·09	3 and 15	−·095
4	−·08	16	−·11	4 and 16	−·095
5	−·10	17	−·08	5 and 17	−·09
6	−·07	18	−·06	6 and 18	−·065
7	+·01	19	+·01	7 and 19	+·01
8	+·09	20	+·06	8 and 20	+·075
9	+·11	21	+·11	9 and 21	+·11
10	+·14	22	+·16	10 and 22	+·15
11	+·16	23	+·10	11 and 23	+·13
12	+·14	0	+·08	12 and 0	+·11
13	+·09	1	+·06	13 and 1	+·075

The differences between the deflections at opposite points of the hour-circle at this station have so much the character of mere accidental irregularities that we may venture for greater simplicity to combine them, as is done in the two last columns of Table V., and to represent the deflections thus combined by a formula of fewer terms, in which a, the hour-angle (reckoned from eighteen hours), is multiplied by 30° instead of 15°. The formula is as follows:—

$$\Delta_s = +\cdot02625 + \cdot1259 \sin(a + 317^\circ\ 51') + \cdot01047 \sin(2a + 25^\circ\ 57') - \cdot0083 \sin(3a + 5^\circ\ 43') + \cdot0118 \sin(4a + 287^\circ\ 48') + \cdot00235 \sin(5a + 291^\circ\ 26') + \cdot0029 \cos 6a.$$

The lunar diurnal variation at St. Helena is, as at Toronto, a double progression, having two westerly extremes at nearly opposite points of the hour-circle, and two easterly extremes also at nearly opposite points. The amounts of extreme deflection, both easterly and westerly, are considerably less than at Toronto, but exhibit the same peculiarity of the extreme elongations in the one direction being about one-third greater than the extreme elongations in the other direction; at Toronto, the easterly elongations are the greater, at St. Helena the westerly. But a still more remarkable distinction in the lunar variation at the two stations respects the lunar hours at which the elongations respectively occur. At Toronto, as we have seen, the two easterly extremes coincide nearly with the epochs of the moon's superior and inferior culminations, and the westerly extremes nearly with the quadrantal hours of 6 and 18. At St. Helena, on the other hand, neither the times of the easterly nor of the westerly elongation coincide either with the culminations or with the hours of 6 and 18; and are so far and so systematically distant from those hours as to leave no doubt of the reality of their differences from them. The above formula gives for the times of extreme easterly deflection at St. Helena $3^h\ 24^m$ and $15^h\ 24^m$; and for the times of extreme westerly deflection $10^h\ 6^m$ and $22^h\ 6^m$; and the four hours at which the lunar variation disappears, or becomes 0, are $2^h\ 0^m$, $6^h\ 53^m$, $14^h\ 0^m$ and

$18^h\ 53^m$. The mean amount of extreme easterly elongation is $4''{\cdot}48$, and of extreme westerly elongation $6''{\cdot}43$, making the whole range of variation in a lunar day due to the moon's influence $10''{\cdot}91$.

The question of any possible difference existing in the lunar diurnal variation at St. Helena when the moon is at her extreme distance on either side of the equator, has been examined in the same manner as has been already described in pages 554 and 555 in the case of the Toronto observations, and with the same general result. The lunar diurnal variation appears to be the same whether the moon is N. or S. of the equator (or in this case N. or S. of the station). The amount of the lunar diurnal variation being altogether less at St. Helena than at Toronto, small irregularities naturally appear larger in proportion. The number of days included in the comparison is also less, extending through five years only instead of six.

Hobarton.—The hourly observations at Hobarton which have been employed for this investigation are from January 1843 to December 1847 inclusive, comprehending five years; their number is 36,820, of which 3242 have been omitted as disturbed, being found to differ from the monthly mean at the same hour to an amount equalling or exceeding 3·4 scale divisions, or $2'{\cdot}41$ in arc.

The deflections are those of the north end of the magnet, and the signs imply, as at each of the other two stations, an easterly deflection when the − sign is used, and a westerly deflection when the + sign is used. The value of a scale division at Hobarton was $0'{\cdot}71$. The deflections are exhibited in the following Table, which may be advantageously compared with the similar table at Toronto, page 552.

TABLE VI.

Lunar hours.	Deflections.	Lunar hours.	Deflections.	Lunar hours.	Deflections.	Lunar hours.	Deflections.
	Scale divisions.		Scale divisions.		Scale divisions.		Scale divisions.
23	+·08	6	−·05	10	+·05	18	−·08
0	+·15	7	−·06	11	+·14	19	−·15
1	+·19	8	−·10	12	+·25	20	−·15
2	+·24	9	−·01	13	+·33	21	−·12
3	+·30			14	+·31	22	−·02
4	+·21			15	+·24		
5	+·07			16	+·15		
				17	+·01		

This table exhibits, as at Toronto and St. Helena, a double progression, having two westerly and two easterly extremes at nearly opposite points of the hour-circle; and still showing, as at the other two stations, the phænomenon of the elongations in one direction being considerably greater than in the other direction. At Hobarton, as at St. Helena, the westerly elongations are the greatest, whilst at Toronto the easterly are greatest. The hours of extreme elongation differ both from those at Toronto and those at St. Helena. The values in Table VI. may be represented by

the following formula, in which a, the hour-angle reckoned as before from eighteen hours, is multiplied by 15°.

$$\begin{aligned}\Delta_{,}= &+\cdot0825-\cdot0487\sin(a+39^\circ\ 15')-\cdot2056\sin(2a+38^\circ\ 00')\\ &+\cdot0173\sin(3a+339^\circ\ 41')-\cdot0161\sin(4a+68^\circ\ 56')\\ &-\cdot0187\sin(5a+274^\circ\ 54')+\cdot0083\sin(6a+323^\circ\ 08')\\ &-\cdot0098\sin(7a+52^\circ\ 26')+\cdot0076\sin(8a+19^\circ\ 06')\\ &-\cdot0023\sin(9a+334^\circ\ 32')+\cdot0015\sin(10a+323^\circ\ 08')\\ &-\cdot0100\sin(11a+274^\circ\ 44')-\cdot0025\cos 12a.\end{aligned}$$

This formula gives for the times of extreme westerly deflection $2^h\ 52^m$ and $13^h\ 21^m$, the amounts being $12''\cdot8$ and $14''\cdot9$; and for the times of extreme easterly deflection $8^h\ 0^m$ and $19^h\ 30^m$, the amounts being $4''\cdot3$ and $6''\cdot7$. The means of the two times of westerly elongation are $2^h\ 06^m\cdot5$ and $14^h\ 06^m\cdot5$, and of the amounts, $13''\cdot85$. The means of the two times of easterly elongation are $7^h\ 45^m$ and $19^h\ 45^m$, and of the amounts, $5''\cdot5$. The total range of the lunar variation is $(13''\cdot85+5''\cdot5=)19''\cdot35$.

The times at which the deflection due to the moon's action is 0 are $5^h\ 27^m$, $9^h\ 06^m$, $17^h\ 05^m$, and $22^h\ 09^m$. These times taken in pairs are equidistant from two points on the moon's hour-circle, of which one is $1^h\ 48^m$ after her upper culmination, and the other is $1^h\ 05^m\cdot5$ after her lower culmination.

General Remarks.—We learn from the results which have been thus stated, that the existence of a lunar diurnal variation in the magnetic declination is shown at each of the three stations, Toronto, St. Helena, and Hobarton, and that it has the same general character at each, viz. that of a double progression in the lunar day with two easterly maxima at nearly opposite points of the hour-circle, and two westerly maxima also at nearly opposite points of the hour-circle. The extreme elongations in the one direction are not at precisely opposite points of the hour-circle at any of the three stations, nor have the amounts of the two elongations, which take place in the same direction, always precisely the same value, but the slight inequalities in these respects are within the limits which might be occasioned by accidental irregularities. It is otherwise, however, with the difference in amount between the easterly and westerly deflections, which exhibit at each of the three stations a disparity too great to be ascribed to accident. At Hobarton and St. Helena the westerly elongations have the largest values, and at Toronto the easterly (speaking always of the north end of the magnet). At Toronto and St. Helena the lesser elongations are about two-thirds of the amount of the larger; at Hobarton the disparity is still greater; to obtain the exact proportions with a sufficient degree of assurance would require, no doubt, a longer period of observation.

The difference which presents itself at the different stations, in the times of the occurrence of the extreme easterly and westerly elongations of the variation, is a very important feature in its bearing on the questions which must arise in regard to the

nature and character of the lunar magnetic influence. At Toronto the easterly elongations take place almost exactly at the hours of upper and lower culmination, and the westerly elongations at the intermediate hours of 6 and 18. At Hobarton the westerly elongations (which, as Hobarton and Toronto are in opposite hemispheres, may be regarded as corresponding to the easterly at Toronto) take place (speaking generally) about two hours after the upper and lower culminations, and the easterly elongations (corresponding to the westerly at Toronto) about two hours after the lunar hours of 6 and 18. At St. Helena, which is in the same hemisphere with Hobarton, the westerly elongations occur about two hours before the culminations, and the easterly extremes about two hours, or rather more, before the hours of 6 and 18. The differences in this respect at the different stations are a feature of such essential theoretical importance, that I should have delayed the presentation of this paper until it could have been accompanied by a similar investigation into the results obtainable from the five years of hourly observations printed in the first volume of the Magnetical Observations at the Cape of Good Hope, had I not the expectation that that portion of the task may be undertaken by Mr. PIERCE MORTON, who now conducts the magnetical observatory at that station.

The entire amount of the variation, reckoned from one extreme elongation to the other, is at Toronto about $27''$; at Hobarton about $20''$; and at St. Helena about $10''$. The terrestrial force, retaining the magnet suspended horizontally in its mean position, may be taken approximately at Toronto at 3·54, at Hobarton at 4·51, and at St. Helena at 5·57. It may be desirable also to state that the amount of the magnetic Declination at the period under consideration was, at Toronto $1° 34'$ West, at Hobarton $9° 55'$ East, and at St. Helena $23° 25'$ West.

The number of observations which have contributed to these results (exclusive of those omitted on account of disturbance) are, 38,398 at Toronto, 33,771 at St. Helena, and 33,578 at Hobarton; making, altogether, 105,747 observations.

With the exception of marking the observations for the lunar hours to which they most nearly correspond, and the calculation of the constants in the formulæ at Toronto and St. Helena, the processes to which the observations received from the stations have been submitted have been executed by Mr. JOHN MAGRATH, principal Clerk in the Woolwich Office, assisted by Sergeant CHARLES ORGAN and Corporal MATTHEW M'ILROY, of the Royal Artillery, employed in the office as military clerks.

Woolwich, November 16, 1853.

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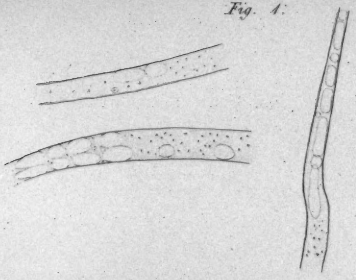

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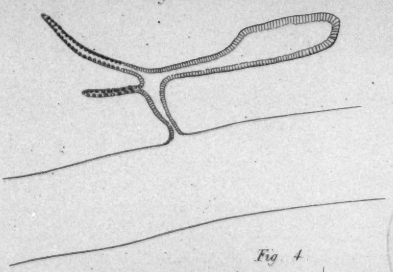

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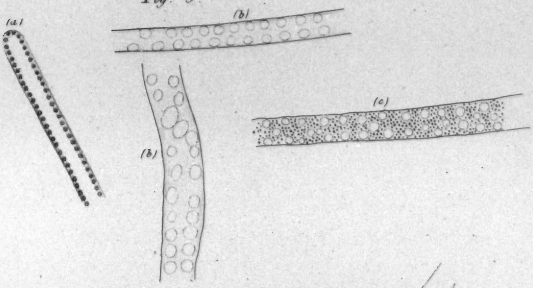

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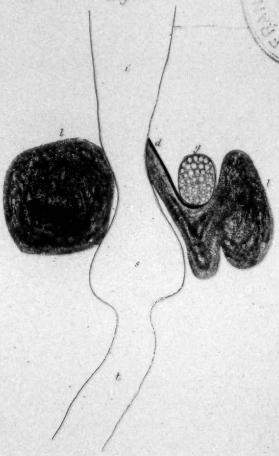

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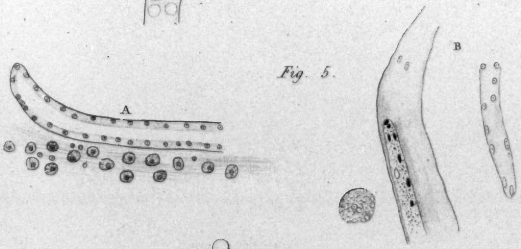

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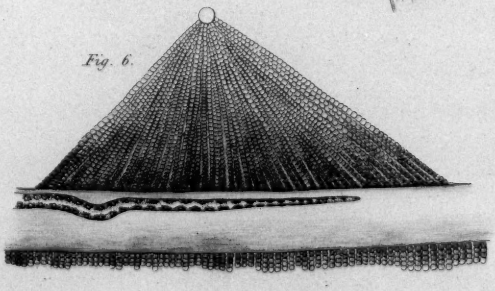

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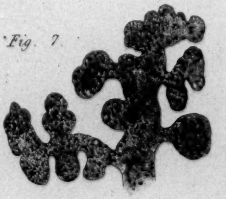

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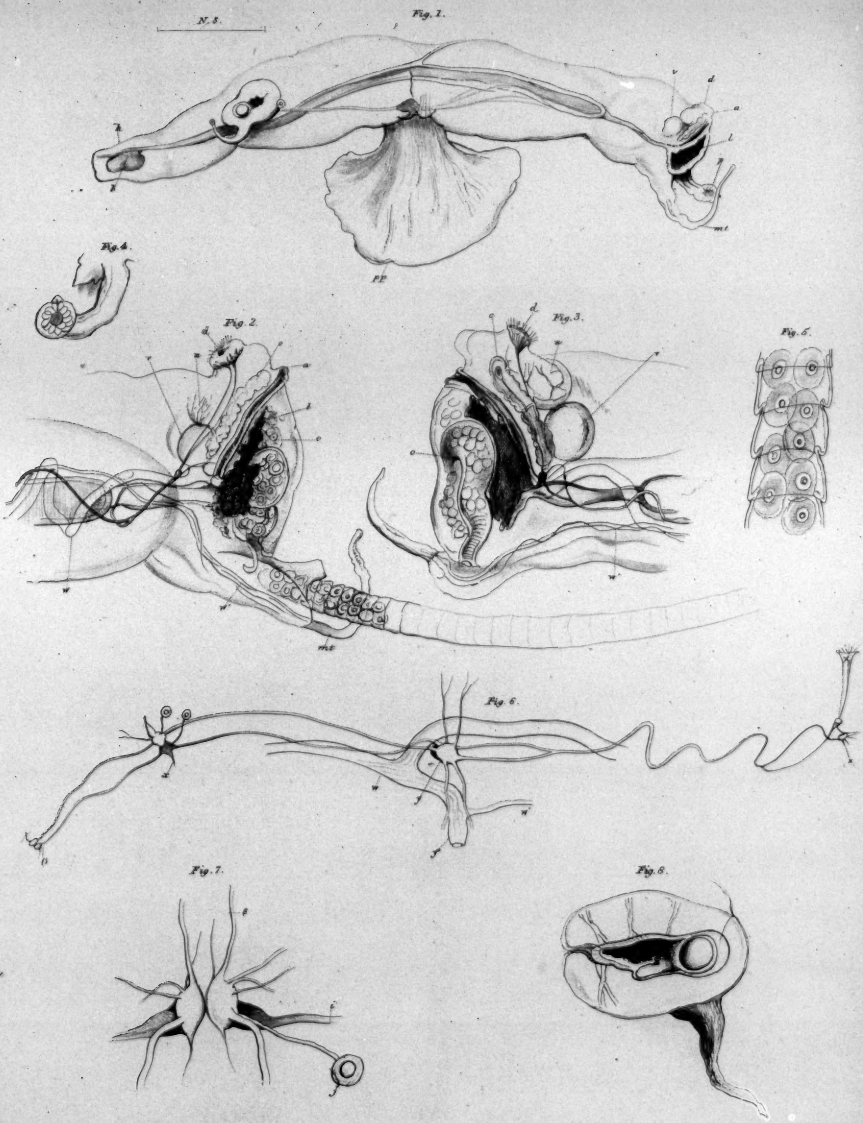
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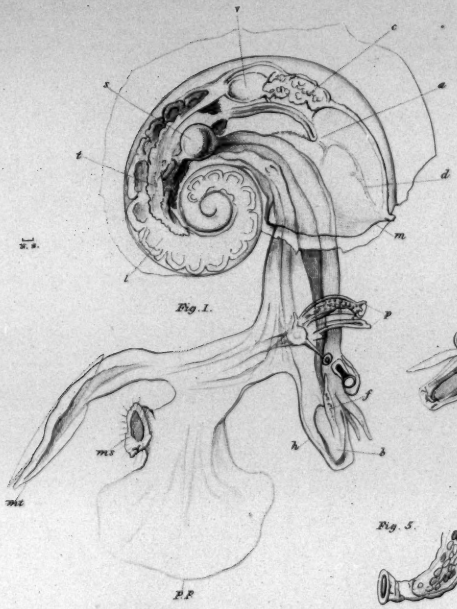

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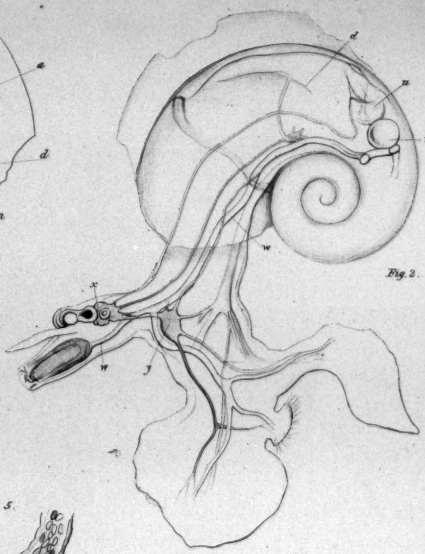

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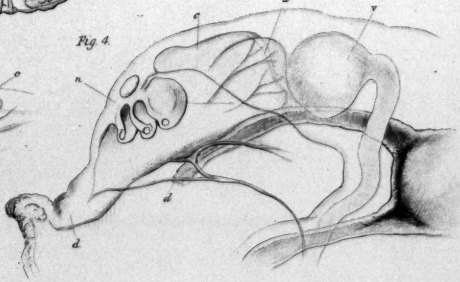

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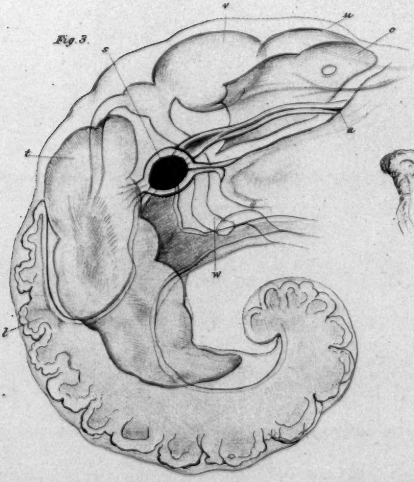

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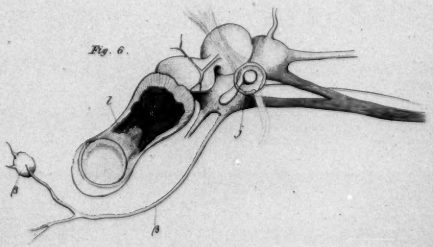

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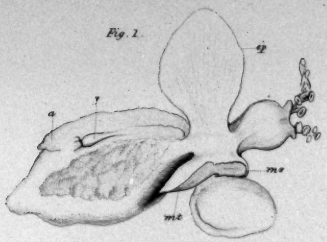
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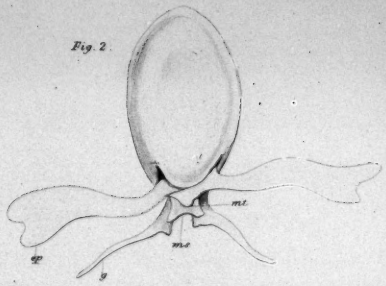
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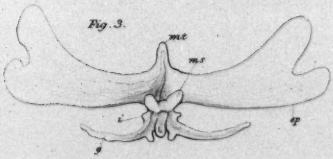
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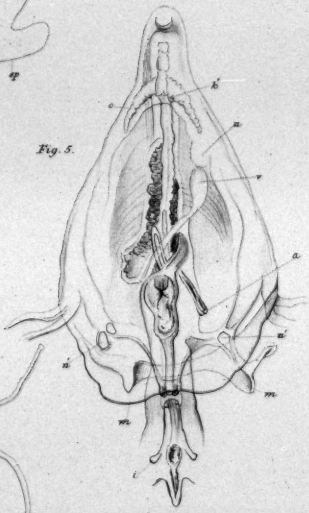
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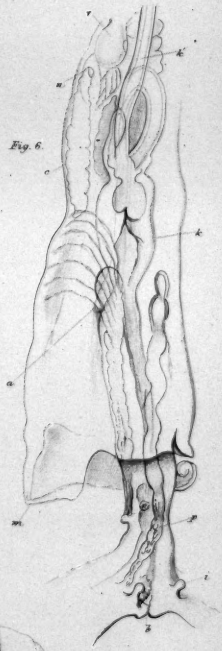
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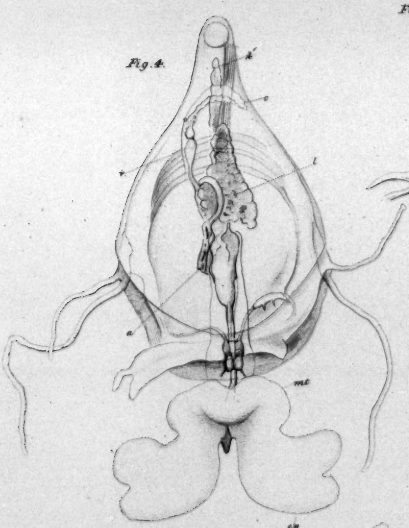
r
k
u
c
k
a
m
p
i
z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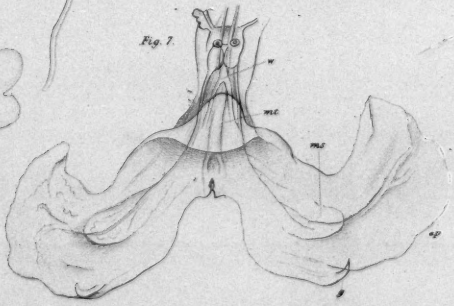
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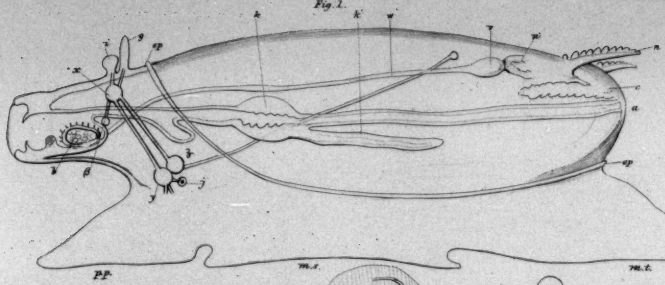
mt
ms
sp
y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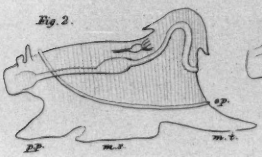
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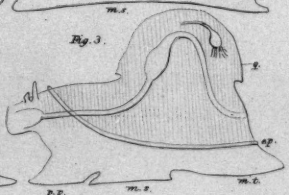
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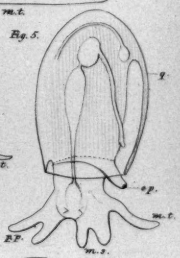
m.s.
m.t.

Fig. 4.
pp.
m.s.
m.t.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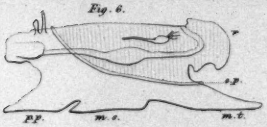
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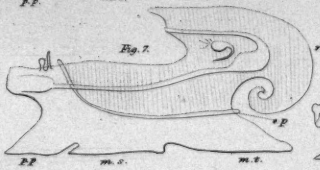
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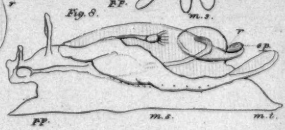
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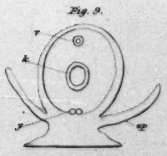
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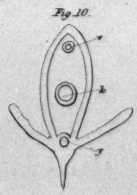
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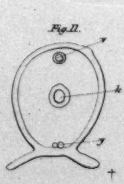
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Fig. 12.

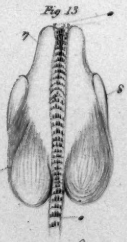


Fig. 13

Fig. 14.

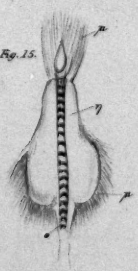
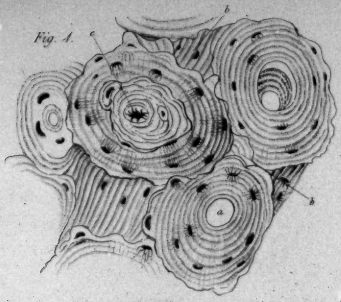


Fig. 15.

Fig. 16.

Fig. 17.

Fig. 1.



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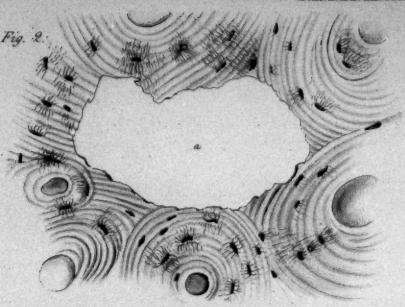

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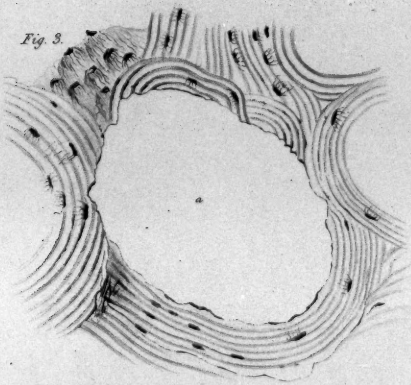

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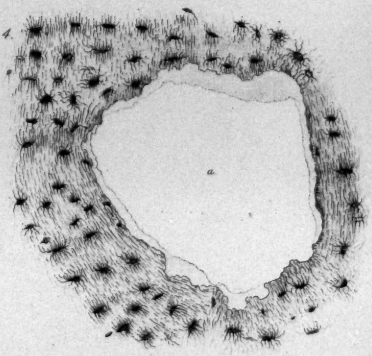

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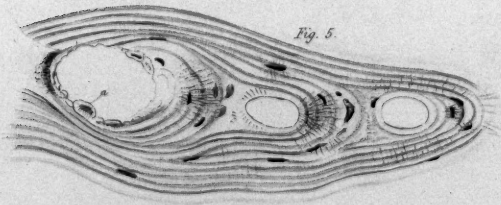

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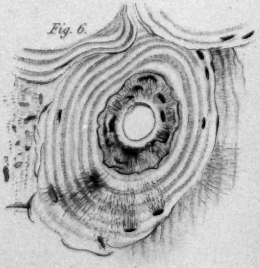

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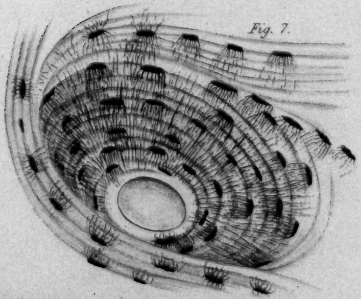

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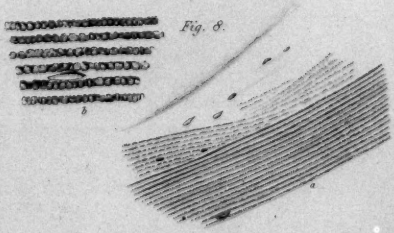
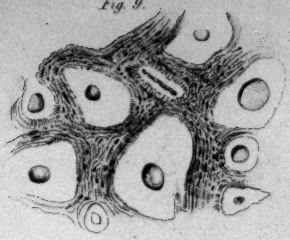


J. Basire, sc.

Fig. 9.



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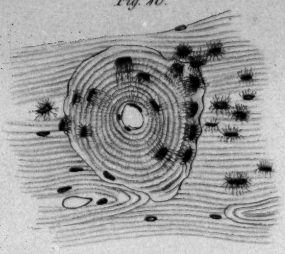

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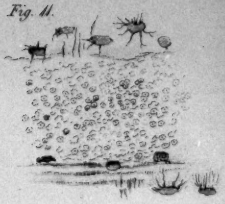

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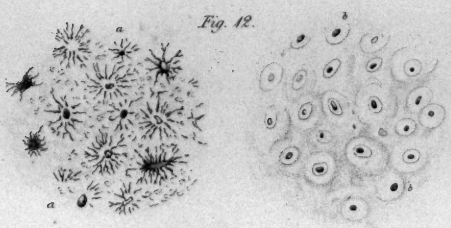

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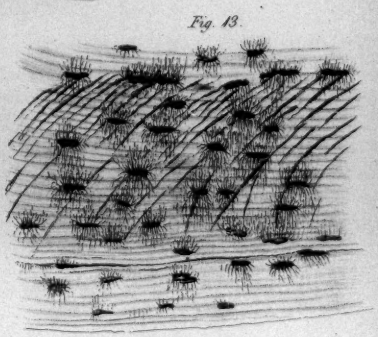

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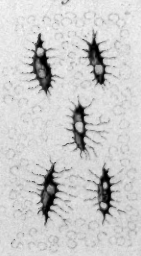


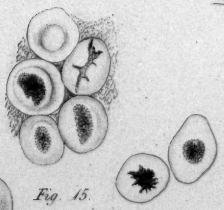
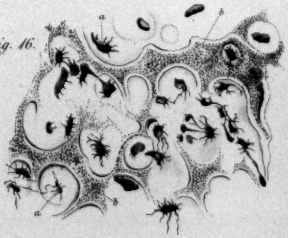


Fig. 15.

Fig. 16.



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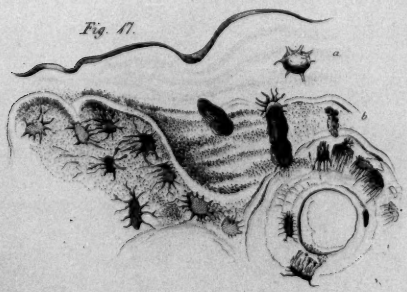

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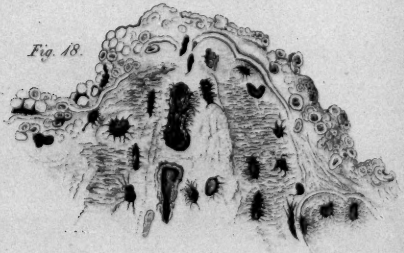
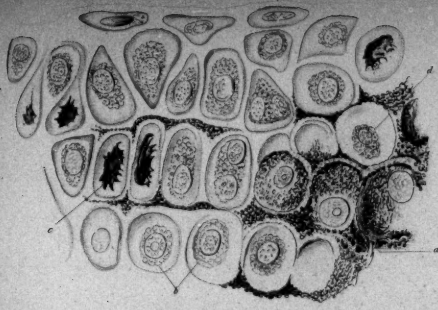


J. Basire sc.

Fig. 19.



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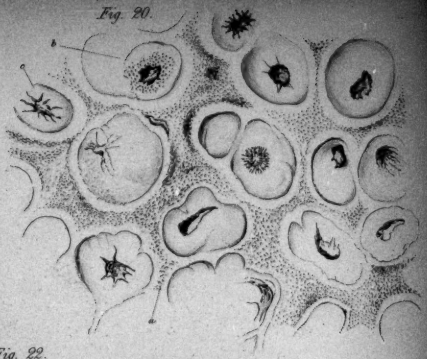

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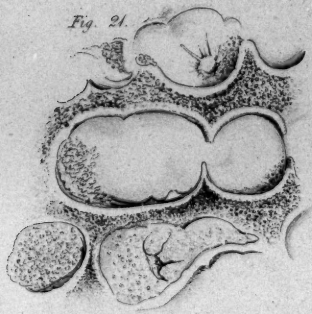
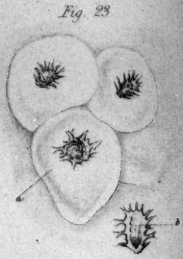


Fig. 22.

Fig. 23.



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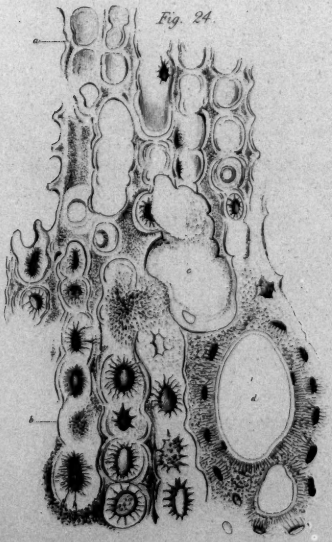

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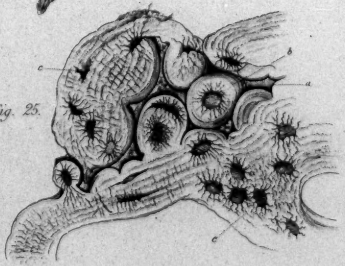
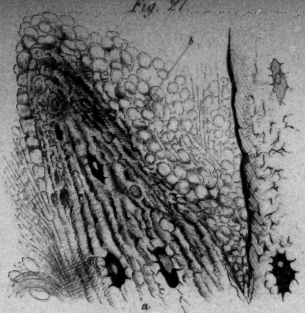


Fig. 26.

Fig. 27.



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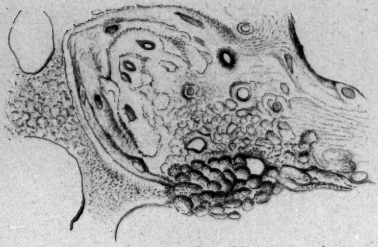

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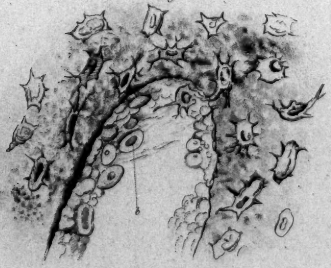

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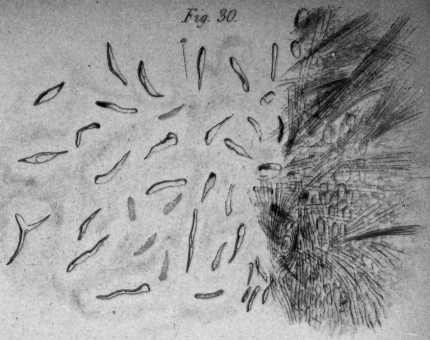
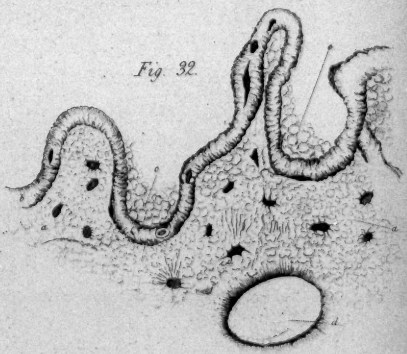


Fig. 31.

Fig. 32.



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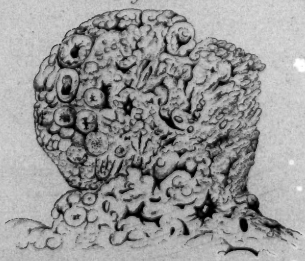

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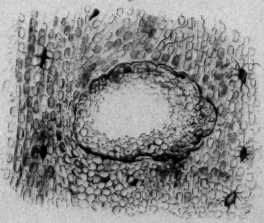
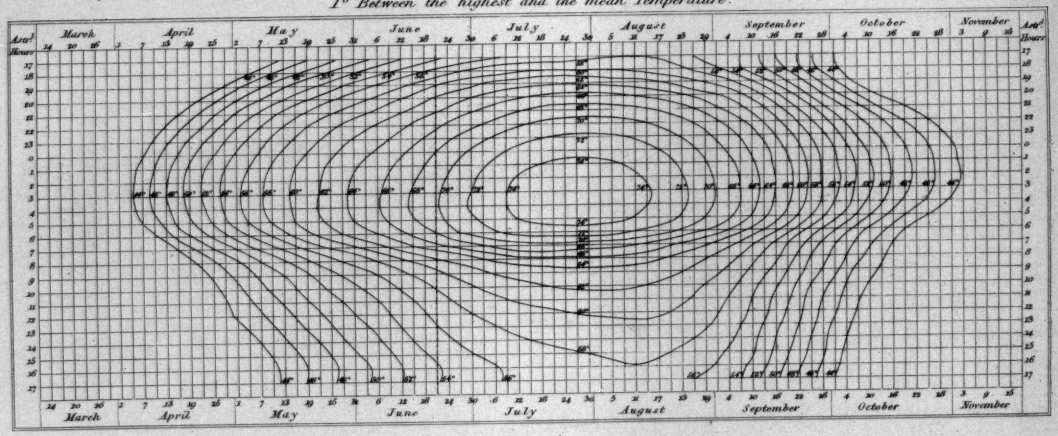
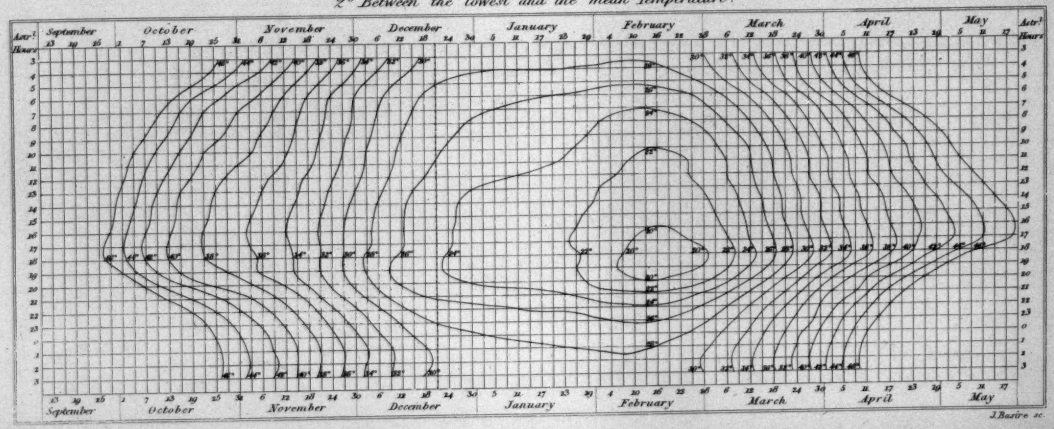


Chrono-Isothermals at Toronto; derived from six years of hourly observations.

1° Between the highest and the mean Temperature.

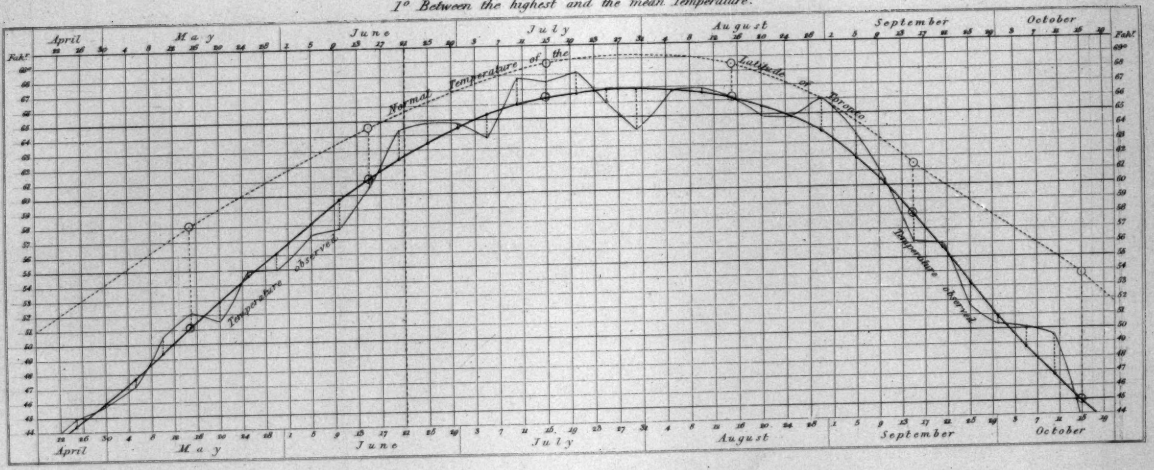


2° Between the lowest and the mean Temperature.

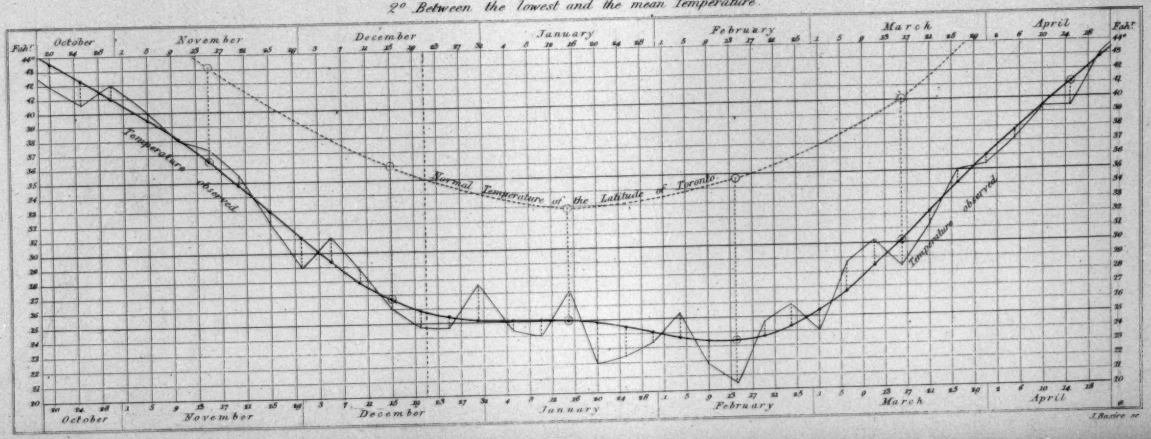


Five-day Means of the Temperature at Toronto from 12 years of Observation.

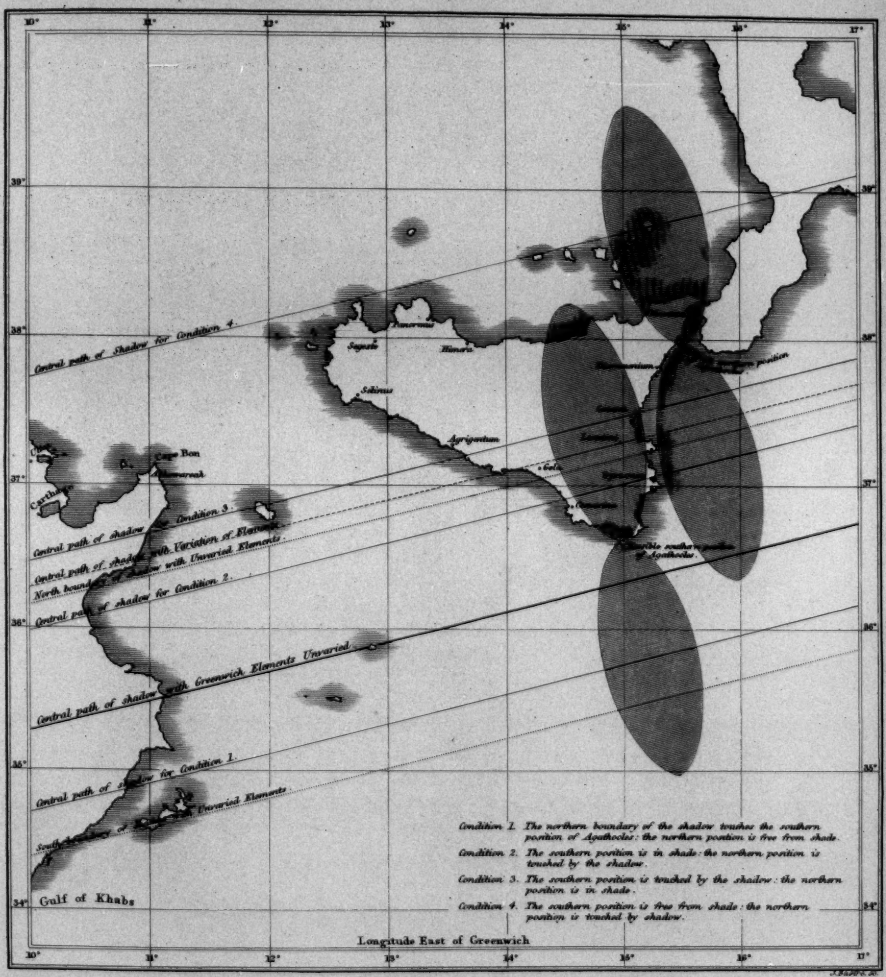
1° Between the highest and the mean Temperature.



2° Between the lowest and the mean Temperature.

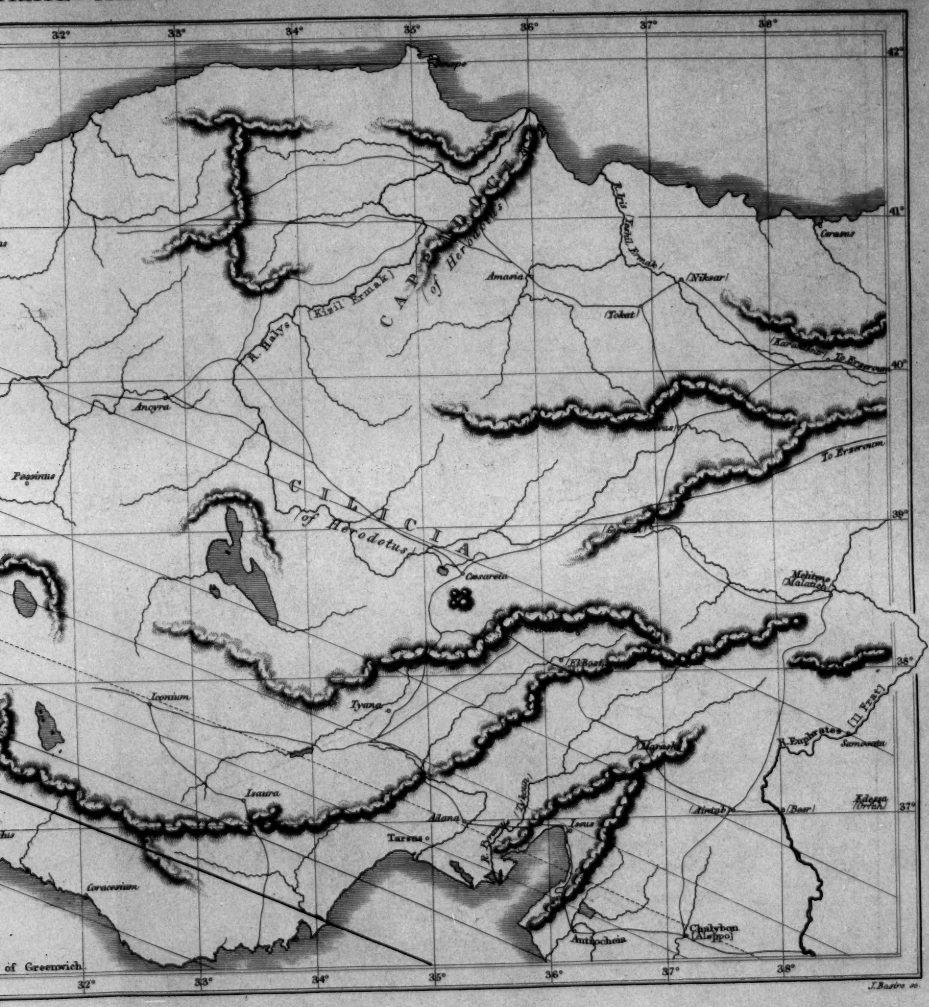


MAP TO ILLUSTRATE THE ECLIPSE OF AGATHOCLES.


Central path of Shadow for Condition 4.
Central path of shadow for Condition 3.
Central path of shadow with Variation of Elements.
North boundary of shadow with Unvaried Elements.
Central path of shadow for Condition 2.
Central path of shadow with Greenwich Elements Unvaried.
Central path of shadow for Condition 1.
South boundary of shadow with Unvaried Elements.
Gulf of Khabs
Cape Bon
Carthage
Sicily
Agrigentum
Selinus
Syracuse
Longitude East of Greenwich
Condition 1. The northern boundary of the shadow touches the southern position of Agathocles: the northern position is free from shade.
Condition 2. The southern position is in shade: the northern position is touched by the shadow.
Condition 3. The southern position is touched by the shadow: the northern position is in shade.
Condition 4. The southern position is free from shade: the northern position is touched by shadow.

MAP TO ILLUSTR

TRATE THE ECLIPSE OF THALES.



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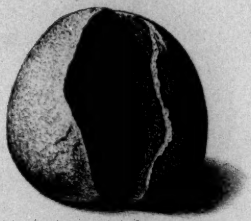


Fig. 6.

Fig. 1*

Fig. 8.

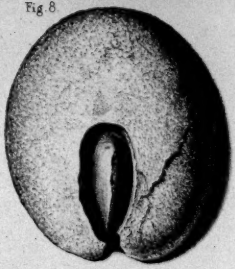


Fig. 5.

Fig. 1*

Fig. 3.

Fig. 4.

Fig. 7.

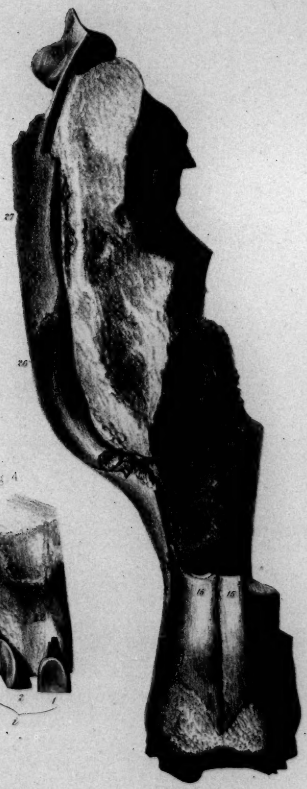
Fig. 2.

from Nature by W. Ross Aldous

Printed by J. Basire

Fig. 1.

Fig. 2.



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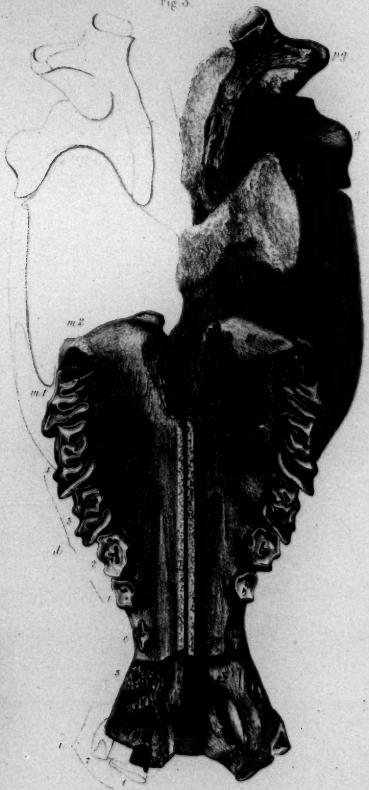


Fig. 4.

Fig. 5.

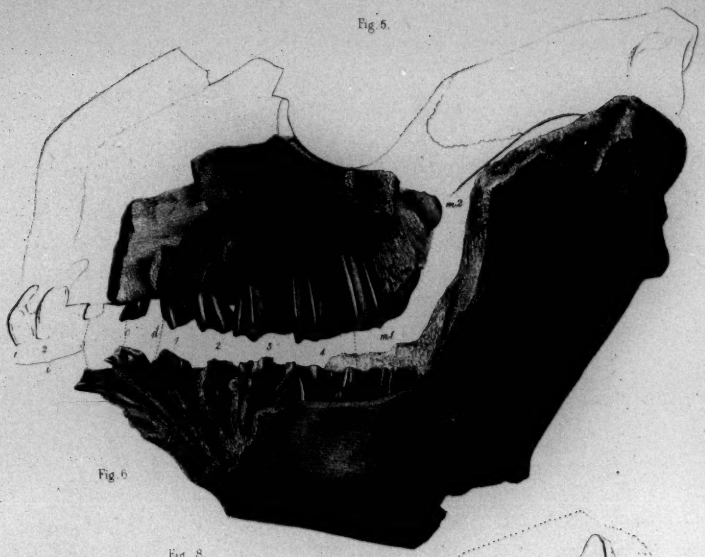


Fig. 6

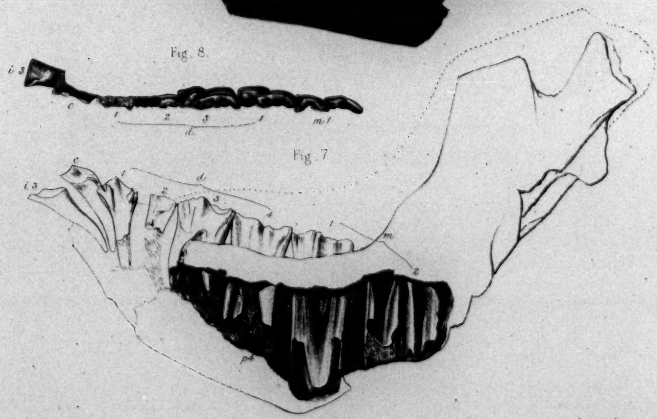


Fig. 8.

Fig. 7.

Fig. 9.

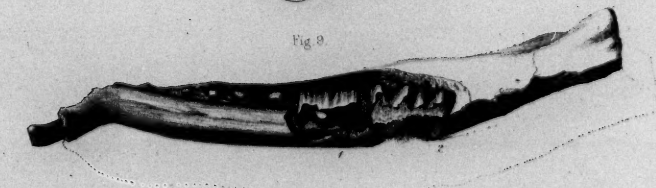
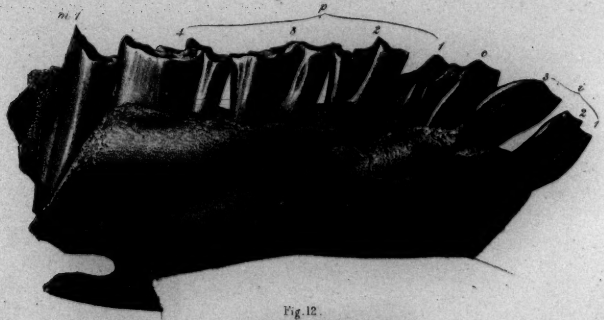


Fig. 10.

Fig. 13.

Fig. 11.



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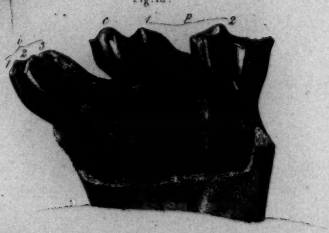


Fig. 14.

Drawn & Lith. by J. Fait.

Printed by J. Haas.

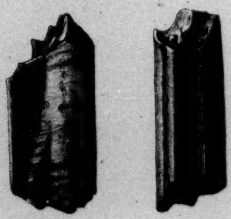
Fig. 15. p 2, r

Fig. 16. p 3, l

Fig. 17. p 4, l

Fig. 18. m 1, r

Fig. 19. m 2.



m 3. Fig. 20.

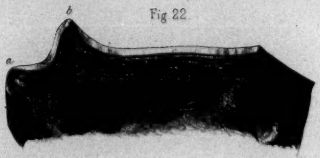


Fig. 22.

Fig. 21.

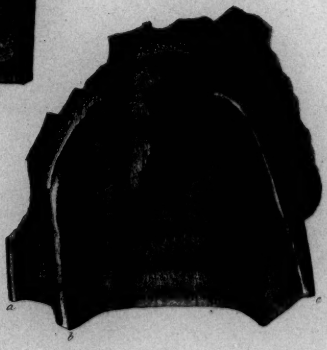
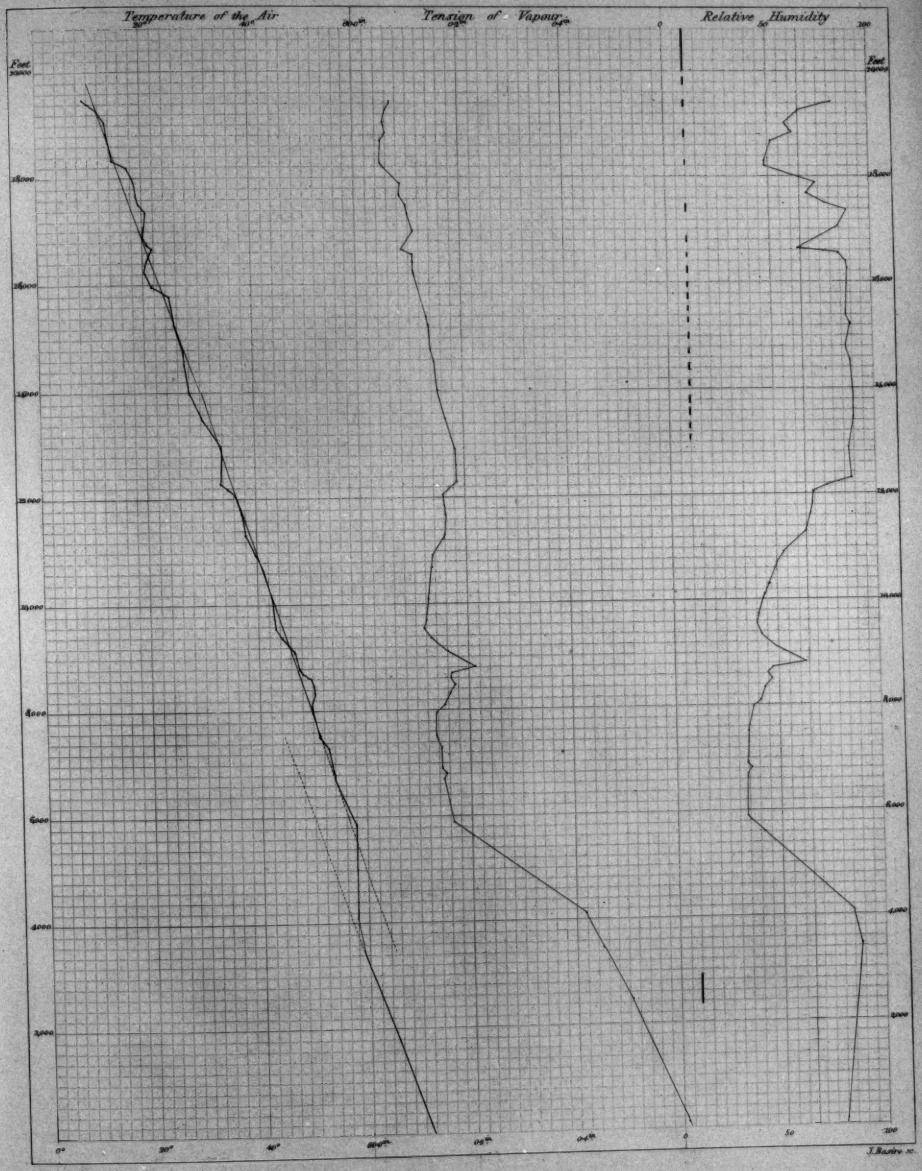


Fig. 23.

J. Dinkel. del. et lith.

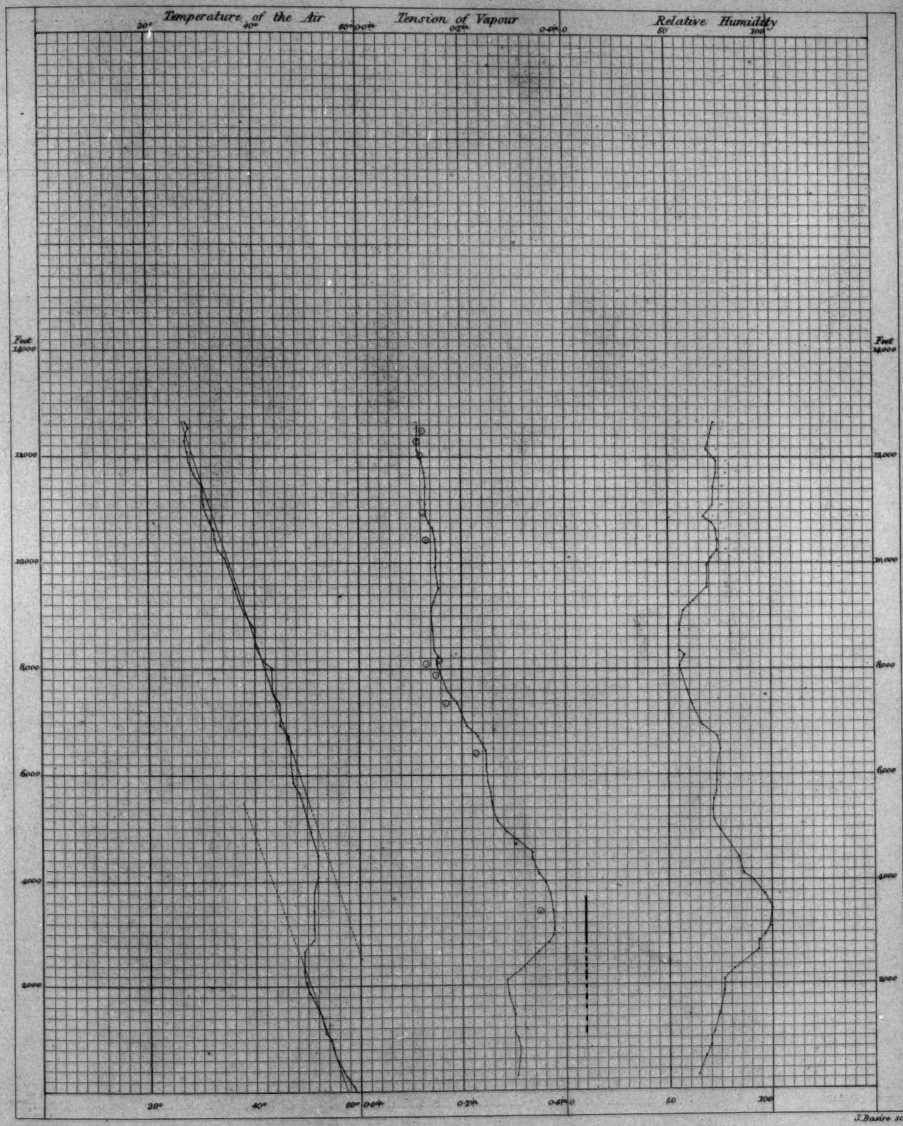
Printed by J. Basire.

Meteorological observations in the Balloon Ascent of August 17th 1852.



Meteorological observations in the Balloon Ascent of August 26th 1852.

Meteorological observations in the Balloon Ascent of October 21.st 1852.



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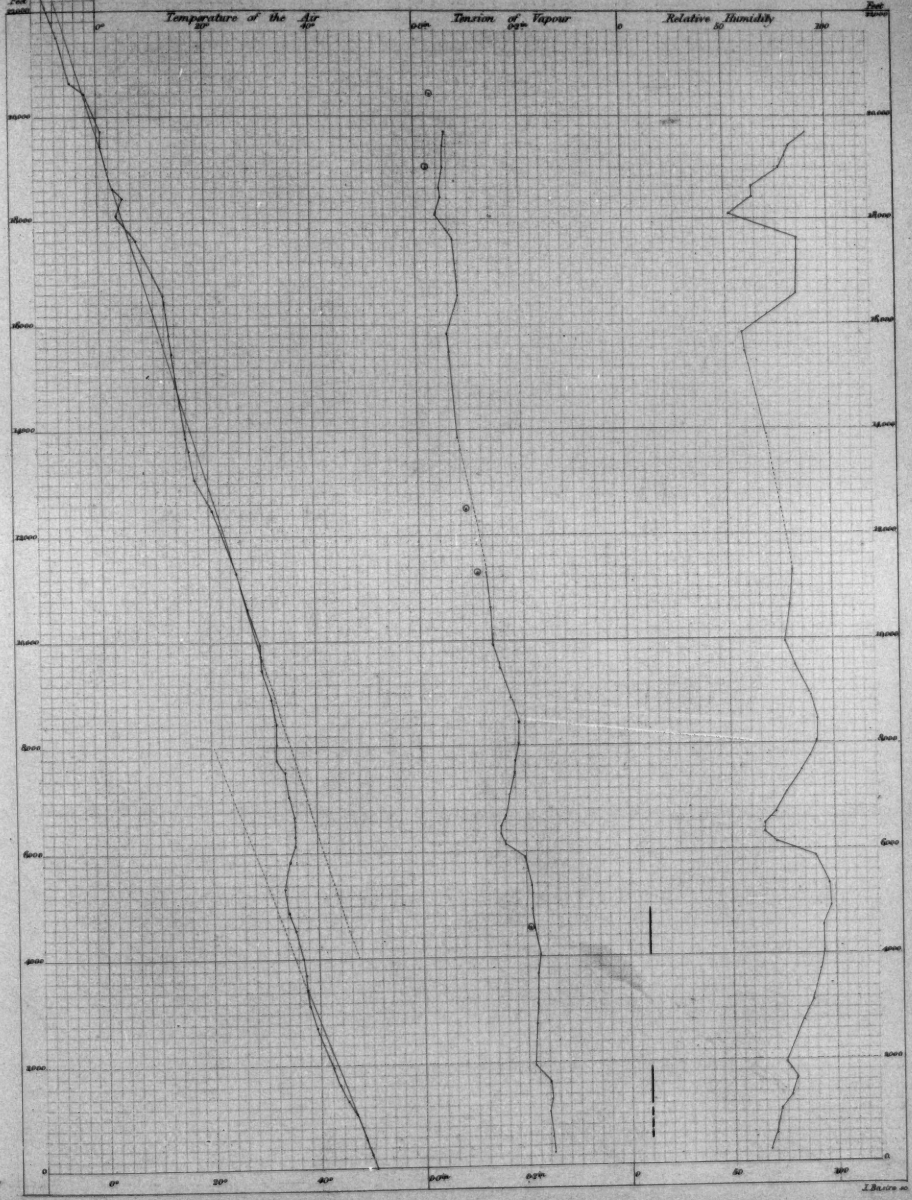
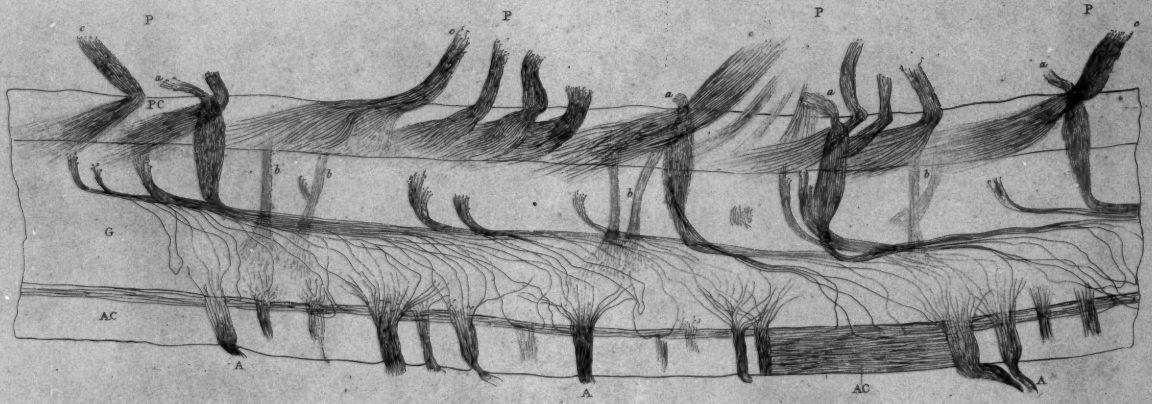
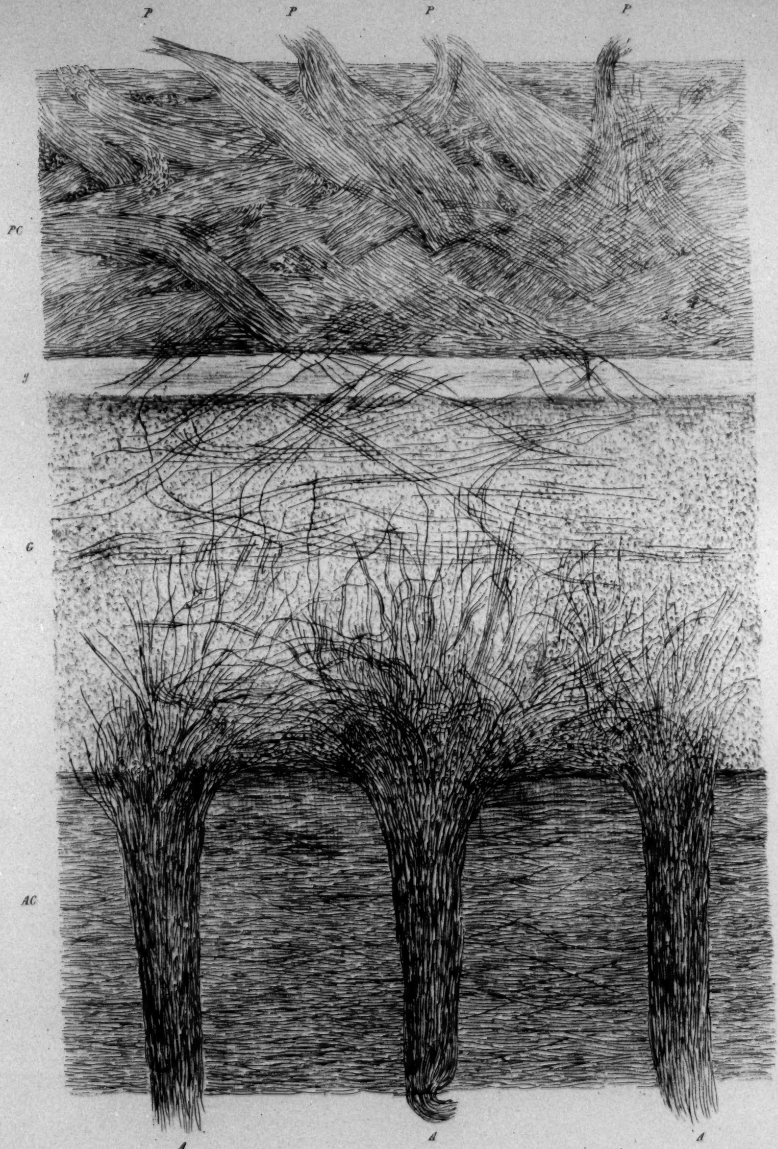




magnified 30 diameters.

Drawn by J.L. Clarke, from nature.

J. Basire sc.



magnified 30 diameters

Drawn by J.J. Clarke, from Nature.

J. Basire lith.

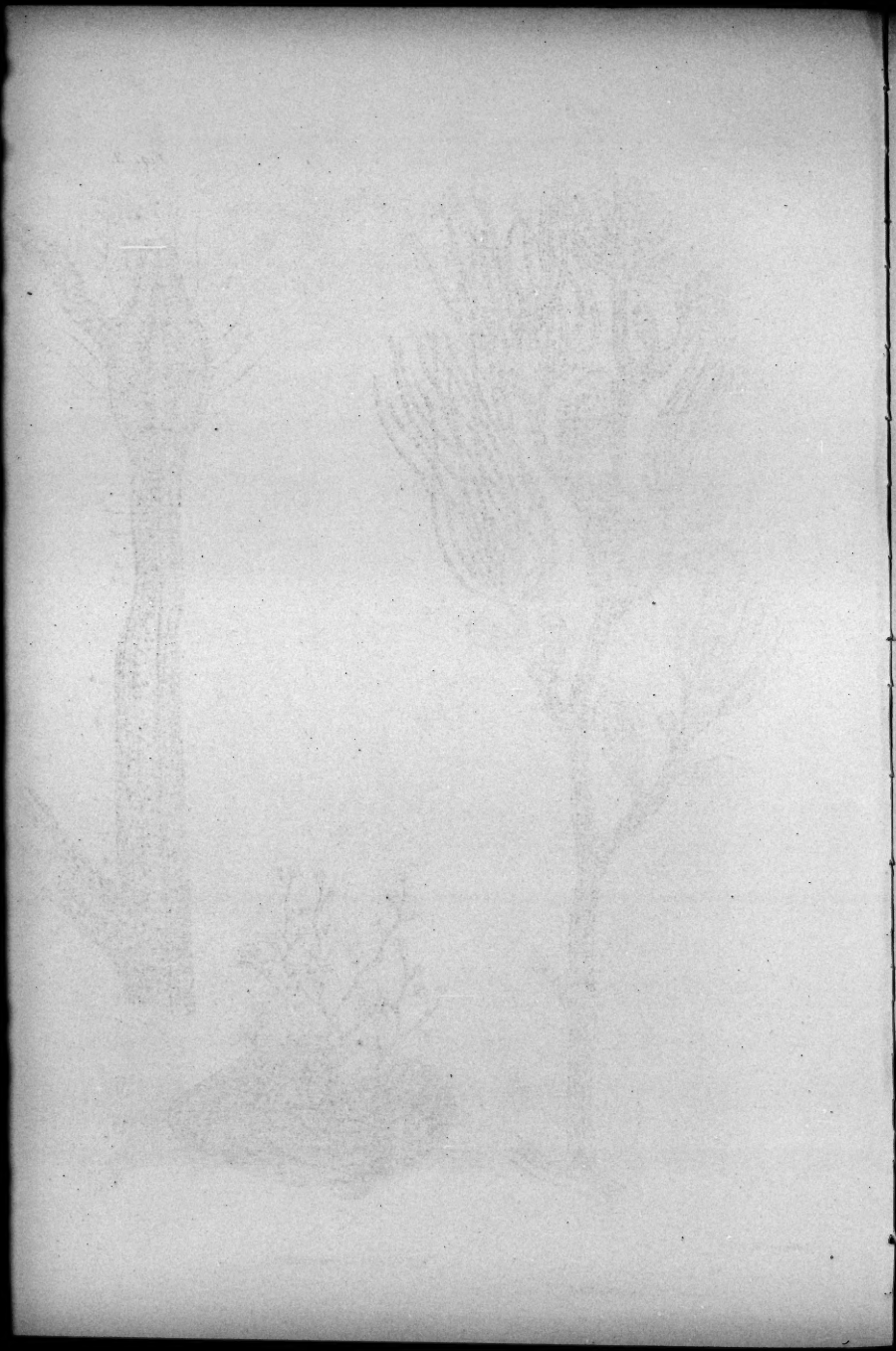
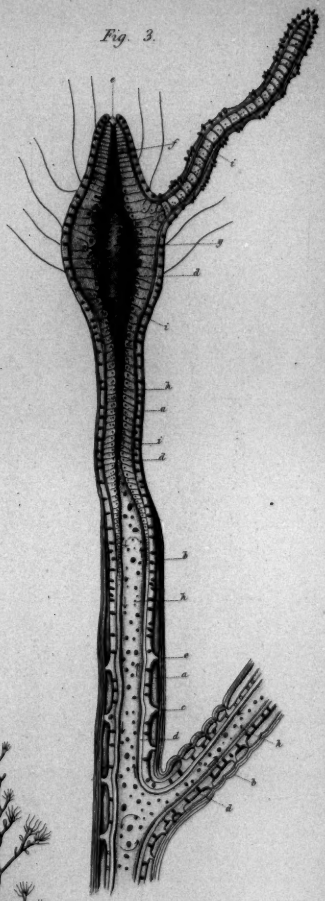


Fig. 2.

Fig. 3.



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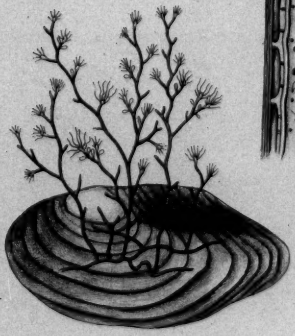

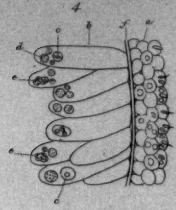
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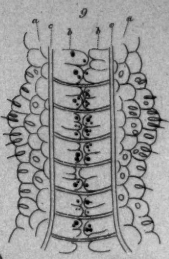
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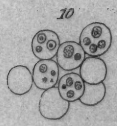
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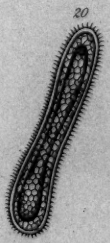
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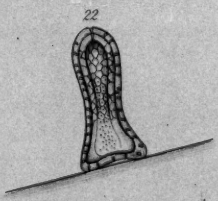
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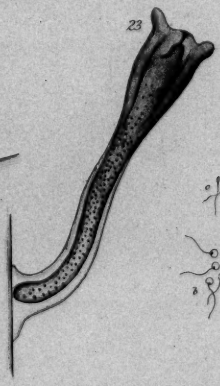
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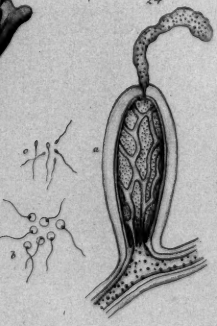
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